# Costs of Financing US Federal Debt Under a Gold Standard: 1791-1933*

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**Abstract**

From a new data set, we infer time series of term structures of yields on US federal bonds during the gold standard era from 1791-1933 and use our estimates to reassess historical narratives about how the US expanded its fiscal capacity. We show that US debt carried a default risk premium until the end of the nineteenth century when it started being priced as an alternative safe-asset to UK debt. During the Civil War, investors expected the US to return to a gold standard so the federal government was able to borrow without facing denomination risk. After the introduction of the National Banking System, the slope of the yield curve switched from down to up and the premium on US debt with maturity less than one year disappeared.

JEL classification: E31, E43, G12, N21, N41.

Key words: Big data, default premium, yield curve, units of account, liquidity premium, gold standard, government debt, pricing errors.

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# 1 Introduction

Early US federal administrations faced high debt levels and interest rates. This led 19th century US Congresses and administrations to adjust monetary, fiscal, and financial arrangements in ways that ultimately led to US government debt becoming a global "safe-asset". Studying these changes can provide useful lessons about how governments can build fiscal capacity. However, doing so requires new data on prices and quantities of US government bonds and statistical tools for handling practical difficulties involved in inferring yield curves from the historical data. This paper takes up these challenges and then reevaluates earlier historical narratives about how *default risk*, *denomination risk*, and *liquidity premia* on US federal debt coevolved along with US monetary-fiscal policies.

For monthly data collected by Hall et al. (2018), we infer term structures of *yields* on US federal bonds with greater than one year to maturity throughout the gold standard period from 1791-1933[1]. Sometimes several currencies circulated simultaneously, including gold and silver coins, greenback paper dollars, and notes issued by state and federally chartered banks. Because gold coins circulated throughout the entire period and were dominant for most of it, we start by estimating yield curves for gold coin denominated US federal debt contracts. We fit a dynamic Nelson and Siegel (1987) model with stochastic volatility and bond specific measurement errors to model our long, historical data set.

We detect several features of gold denominated bond yields during the "gold standard era." First, yields trended downwards throughout the 19th century, with the 10-year gold yield dropping from around 8% in 1800 to around 2% in 1900, and stabilizing around that level until World War I.[2] Second, there were large spikes in yields during 19th century wars. We find substantially larger yields during the Civil War than earlier researchers. Third, yield curves typically sloped downward before the Civil War but then turned slightly positive after the late 1870s.

We use our estimated yield curves to revisit historical narratives about US borrowing costs during the 19th century. In Section 5 we study default risk premia by comparing yields to maturity on gold denominated UK consols to yields-to-maturity on hypothetical gold denominated US consols that promise the same coupon flows as the UK consols. Since UK consols were widely considered to be "the" safe asset during the 19th century

[1]George Washington and Alexander Hamilton introduced a gold standard in 1791 that was theoretically maintained until 1933, at which point Franklin Roosevelt accepted Irving Fisher's advice to deviate from it.

[2]This is consistent with the global super-secular decline hypothesis put forward by Schmelzing (2020) but suggests that, from the turn of the 20th century, US debt started to play a different role than other debt securities.

and since both countries were on a gold standard for most of this period, we interpret the spread as a risk premium on US debt. We find that US debt typically carried a risk premium until 1905, when US yields became persistently lower than UK yields. This indicates that US debt started to have characteristics of a global "safe-asset" well before World War I.

In Section 6 we investigate denomination risk. In February 1862, Congress passed the Legal Tender Act, which authorized the Treasury to issue inconvertible greenback dollars. During and after the Civil War, the federal government issued bonds denominated in both gold and greenback dollars, a situation that enables us to estimate a greenback denominated yield curve and also investors' expectations about the volatile gold-greenback exchange rate. Despite a 60% depreciation in the greenback-to-gold exchange rate during the Civil War, we infer a strong nominal anchor: investors seemed to have anticipated that greenbacks would trade one-for-one with gold dollars after the war. An anticipated appreciation of the greenback kept greenback yields persistently lower than the gold yields. Having that firm nominal anchor helped the Union government raise wartime revenues by printing greenbacks.

Finally, in Section 7, we investigate short term bond premia. During 1862-66, Congress passed four National Bank Acts that constructed a system of federally chartered banks that could issue standardized bank notes backed by long-term U.S. government debt. Their goal was to increase the supply of short term assets and increase banks' demand for long term US federal debt and thereby lower long term yields. Our yield curves contain evidence about consequences of those reforms. We estimate differences between observed and model-implied yields to maturities for bonds with less than one year to maturity.[3] Following others, we interpret this spread as a "short-rate-disconnect" or "liquidity premium" on short term bonds. We find that for most of the 19th century short-maturity government debt traded at a 0.25 to 0.5 percentage point premium, which indicates that money-like assets were scarce and so earned a "liquidity" premium. But premia on short term debt disappeared during the high tide of the National Banking Era from 1885 until 1917. This indicates that the short rate disconnect co-varied with regulatory changes.

To specify and estimate a stochastic process for yield curves on US federal bonds, we faced several technical challenges. Our data set is sparse along the cross-section dimension. We acknowledge this by adopting a dynamic version of a tightly parameterized Nelson-

[3]Because our model was estimated using bonds with maturity greater than 1 year, this spread represents an "out-of-sample" fit at the short end of the yield curve.

Siegel yield curve model proposed by Diebold and Li (2006). Another challenge is that 19th century US federal bonds sometimes included call options and other features that gave lenders and/or the Treasury some discretion over maturity dates and conversions. Our inference procedure assumes that agents priced bonds under perfect foresight about those discretionary contract features. To prevent that assumption from unduly influencing our inferences, we include bond-specific idiosyncratic pricing errors. We compare our approach to some alternatives in Payne et al. (2023a) and discuss its advantages for handling historical data sets.

We face additional challenges when estimating the greenback denominated yield curve during the period 1862-1878. Some bonds promised all payments in gold while other bonds promised all payments in greenbacks. Our statistical challenge is that there are few greenback paying bonds and most of them are short term. In order to estimate the greenback yield curve, we extend our procedure to use information on greenback paying bonds, the gold-to-greenback exchange rate, and our gold yield curve. We incorporate a state-space model of the gold-to-greenback exchange rate. We then estimate the joint process of the greenback yield curve and anticipated gold-to-greenback exchange rates *conditional* on our baseline gold yield curve estimates. This approach could be applied in other settings where a government issues bonds denominated in a mixture of currencies.

Properties of our data set induce us to estimate yield curves in different ways than modern researchers do. Because the treasury now issues more than enough bonds to pin down a yield curve, it is possible for modern researchers to get tight estimates of yield curves by focusing on a convenient subsample of bond prices. Because our data set typically contains prices of so few bonds at each date, we want to extract information from as many bonds in our data set as we possibly can, including bonds with peculiar and troublesome features that include their callability, convertibility, and so on. Doing this requires that we adapt and augment prominent contemporary statistical models to account for peculiar characteristics of our sparse data set.

RELATED WORK In the spirit of Friedman and Schwartz (1963), we present a narrative history supported by data and statistics. Other recent work has compiled international historical interest rate series and examined long-term trends (e.g., Shiller (2015), Hamilton et al. (2016), Jordà et al. (2019), Schmelzing (2020), Officer and Williamson (2021), Chen et al. (2022)). Data in those studies had limited coverage of US federal yields. Instead, they use a commercial paper rate as a "short interest rate" and a "long market rate" from Homer and Sylla (2004) that combines yields-to-maturity on US federal bonds pre-Civil

War with yields-to-maturity on New England Municipal bonds and corporate bonds post-Civil War. By estimating the full yield curve on US federal bonds, this paper opens up exciting new questions about historical trends in government financing costs.

Technically, our work is related to Svensson (1995), Dahlquist and Svensson (1996), Cecchetti (1988), Annaert et al. (2013), Andreasen et al. (2019), Diebold and Li (2006), Diebold et al. (2006) and Diebold et al. (2008) who, like Gürkaynak et al. (2007) and ourselves, implement versions of the parametric yield curve model of Nelson and Siegel (1987). A "Dynamic Nelson-Siegel Approach", studied in detail in Diebold and Rudebusch (2013), makes assumptions about yield curve parameters' time-variation in order to improve the model's forecasting ability. We make assumptions about yield curve parameters' time-variations in order to pool information across time periods.

Our analysis of events during the 1862 to 1879 greenback period revisits issues presented in landmark studies of Mitchell (1903, 1908), Friedman and Schwartz (1963), Roll (1972), and Calomiris (1988). Our analysis of the national banknote issuance puzzle revisits work by Cagan (1965), Cagan and Schwartz (1991), Champ et al. (1994), Champ and Wallace (2003), Champ (2007), and Calomiris and Mason (2008).

Our comparison between US and UK yields is complementary to Chen et al. (2022), which connects the emergence of US debt as a safe asset to the fiscal positions of the UK and the US. Our study of yield curve slopes is complementary to a large literature surveyed by Gürkaynak and Wright (2012) that tests the expectations hypothesis and studies the predictive power of the yield curve.

Outline Section 2 describes data and provides historical context. Section 3 outlines how we parameterise gold dollar yield curves and delineates our econometric strategy. Section 4 discusses some stylized facts about the "gold standard era" from 1791 to 1933 and compares our estimates to other available historical interest rate series. Section 5 compares historical US and UK yields. Section 6 discusses statistical inferences about greenback dollar yield curves and gold-greenback dollar exchange rate expectations during and after the Civil War. Section 7 estimates the short-rate-disconnect and revisits the national bank note puzzle. Section 8 concludes.

# 2 Data and Historical Context

We have assembled prices, quantities, and descriptions of all securities issued by the US Treasury from 1776 to 1960.[4] Our bond price data are monthly. When available, we use the closing price at the end of each month. When a closing price is not available, we use an average of high and low prices or an average of bid and ask prices. In Appendix A, we spotlight decisions about our data that we made to prepare the statistical inferences presented in this paper. A key decision is to restrict our sample to bonds with maturity greater than one year. In Sections 3 and 7 we explain why we find this necessary due to potential "liquidity premia" on short term government bonds. In this section, we provide historical context.

*Monetary Policy:* US federal monetary policies varied over our sample. From April 1792 to February 1862, the US dollar was defined in terms of gold and silver (a "bimetallic" system). From 1862-1878, the US government also issued non-convertible greenback dollars that circulated alongside gold coins. The greenback depreciated substantially during the Civil War and did not attain parity with gold until January 1, 1879, when the US Treasury started converting greenbacks into gold dollars one-for-one. Convertibility between gold dollars and US notes at par prevailed until 1933 when President Franklin D. Roosevelt increased the paper price of gold and prohibited private US citizens from holding gold coins. For the purposes of this paper, we consider this the end of the US gold standard.

*US Federal Debt Policy:* Before World War I, the federal government issued bonds infrequently. New bond issues were often small. The US Congress, not the Treasury, designed each government security with the consequence that securities varied over time in terms of their coupon rates, denominations, maturities, units of account, tax exemptions, and call features. Before the 1920s, the federal government occasionally issued customized long term debts, mostly to finance wars, debt reschedulings, and specific infrastructure projects. As a result, between 1776 and World War I, the US Congress only authorized the Treasury to issue a total of approximately 200 distinct securities, with at most 8 distinct ones being authorized in any one year.

Between 1917 and 1939, Congress gradually delegated more and more decisions about designing US debt instruments to the Treasury and the Treasury gradually standardized

[4]The data set (excluding any data taken from non-publicly accessible data sets) is available at the Github repository `https://github.com/jepayne/US-Federal-Debt-Public` and construction methods are explained in Hall et al. (2018).

security design. As a result, from 1920 to 1960 alone, the Treasury issued about 2500 securities with a wide range of maturities. Ultimately, this transformed the market for US Treasury securities into the world's most liquid debt market with a collection of standardized securities at many maturities that allowed a large national debt to be issued and rolled over.

When gold and greenback dollars coexisted between 1862 and 1878, different US Treasury bonds promised payments in different currencies. Some bonds promised all payments in gold (we refer to these as "gold dollar" bonds), while others promised all payments in greenbacks (we refer to these as "greenback dollar" bonds). There was also some important bonds, notably the famous 5-20's, that offered coupons in gold but left ambiguous whether the principal would be paid in gold or greenbacks (in Figure I, we refer to these as "ambiguously" denominated bonds). We exclude the 5-20's from the statistical analysis in this paper.

*Data Challenges:* Figure I depicts the monthly time series for the number of securities with observed prices and times to maturity of all outstanding bonds. There were often fewer than 5 price observations in a given month and no price observations in the late 1830s when the Federal government had no outstanding debt. This means that while we have "big data" in the time series dimension, we have few observations in the maturity dimension in any particular month. This forces us to find ways to repurpose tools from the yield curve estimation literature. In Section 3, we posit a statistical model that lets us learn about yields at all dates simultaneously by pooling information across time periods.

The presence of bonds denominated in different currencies allows us to estimate both gold dollar and greenback dollar nominal yield curves. Because we observe only 9 greenback dollar bonds, however, we estimate the two yield curves differently. In Section 3, we focus on gold dollar bonds and fit a parametric model to obtain our baseline yield curve estimate. By contrast, for greenback paying bonds we do not estimate a parametric yield curve. Instead we model greenback yield curves as a combination of the gold dollar yield curve and a parameterized family of expectations about the appreciation of the gold-to-greenback exchange rate. In Section 6, we outline our estimation strategy that incorporates our baseline gold dollar yield curves, the available greenback paying bond prices, and data on the gold-to-greenback and gold-to-goods exchange rates.

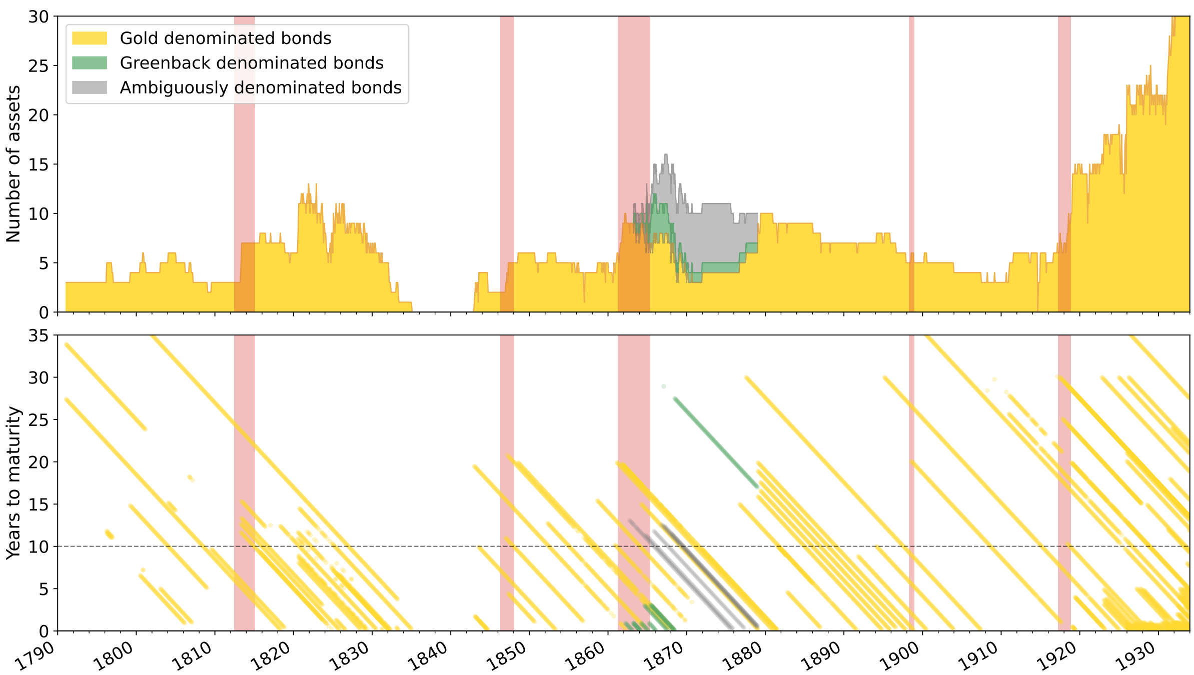


Figure I: Bonds in The Bond Price Set

The top panel depicts the number of securities with observed prices each month. The bottom panel depicts maturities (in years) of observed securities. Darker lines indicate overlapping securities. Red bars correspond to wars.

# 3 Estimating Yield Curves

In order to facilitate comparability of our historical estimates of gold denominated yield curves with modern counterparts we use the popular Dynamic Nelson-Siegel (DNS) model of Diebold and Li (2006) and Diebold et al. (2006) extended with stochastic volatility as in Hautsch and Ou (2008) and Hautsch and Yang (2012).[5] This model allows for dynamic relationships among three factors (that can be thought of as levels, slopes, and curvatures of yield curves), which enables us to learn about yields at all dates simultaneously by pooling information across time periods. This feature of the model is essential to address the cross-sectional sparsity of our data set. Stochastic volatility in the factor dynamics lets the amount of information pooling vary over time. That helps the model accommodate consequences for gold denominated yield curves of wars and important institutional changes.

In a departure from the literature, we add *bond-specific pricing errors* to the observation equation of our model. Throughout our sample, many US Treasury securities had peculiar features such as indefinite maturities associated with call and conversion options. To manage such issues, researchers of the post-WWII sample have usually homogenized their samples by carefully pre-selecting bonds with specific properties. For us, doing that would sacrifice too much information. Furthermore, to us it is a priori unclear which bond characteristics are important for historical bond pricing. Instead, we use bond-specific pricing errors to prevent heterogeneous bond characteristics from distorting our estimates without having to drop too many bonds from our sample.

## 3.1 A Dynamic Nelson-Siegel Model

Let $q_t^{(j)}$ denote the gold price of a government promise to one gold dollar at time $t + j$. We call the sequence $\mathsf{q}_t := \{q_t^{(j)}\}_{j=0}^{\infty}$ a *discount function*. Suppose that at time $t$ bond $i$ promises a sequence of gold dollar coupon and principal payments $\mathsf{m}_t^{(i)} := \{m_{t+j}^{(i)}\}_{j=1}^{\infty}$. We let $p_t^{(i)}$ denote the price of such a coupon-bearing gold dollar bond in terms of gold. We parameterize the gold dollar discount function $\mathsf{q}_t$ by parameterizing the corresponding $j$-period zero-coupon yields defined as $y_t^{(j)} := -\log q_t^{(j)}/j$.

[5]Formally, the models in Diebold and Li (2006) and Diebold et al. (2006) include AR(1) factors whereas we have random walk factors. In our companion paper Payne et al. (2023a) we allow for AR(1) factors but find they do not significantly improve the fit and so are not selected by our information-criterion.

**Assumption 1.** The $j$-period gold dollar zero-coupon yield takes the form

$$y_t^{(j)} = L_t + S_t \left( \frac{1 - \exp(-j\tau)}{j\tau} \right) + C_t \left( \frac{1 - \exp(-j\tau)}{j\tau} - \exp(-j\tau) \right)$$

where $L_t$, $S_t$, and $C_t$ are hidden factors that characterize the level, slope, and curvature of the yield curve at time $t$. The vector $\lambda_t := [L_t, S_t, C_t]'$ follows a drift-less random walk:

$$\lambda_{t+1} = \lambda_t + \Sigma_t^{\frac{1}{2}} \varepsilon_{\lambda,t+1} \tag{3.1}$$

where $\Sigma_t$ is a covariance matrix with $\Sigma_t = \Xi_t \Omega \Xi_t$. Here, $\Omega$ is the time-invariant correlation matrix and $\Xi_t$ is a diagonal matrix with marginal standard deviations $\sigma_t$ that follow:

$$\log \sigma_{t+1} = \log \sigma_t + \Xi_\sigma \varepsilon_{\sigma,t+1} \tag{3.2}$$

where $\Xi_\sigma$ is a positive definite diagonal matrix. Shocks $\{\varepsilon_{\lambda,t}, \varepsilon_{\sigma,t}\}_{t \geq 1}$ are Standard Normal.

This is the yield curve model of Diebold and Li (2006) and Diebold et al. (2006) except for the law of motion of the yield curve factors (3.1). Different from these papers, we assume that the yield curve factors are random walks and the corresponding shock volatilities are time-varying. The matrix $\Sigma_t$ governs the degree of information pooling over time.[6] The closer are two dates to each other, the more correlated are the associated yield curves, with $\Sigma_t$ capturing what "close" means. The limit $\Sigma_t \to 0$ corresponds to *complete pooling*: here the yield curve is assumed to be fixed over time. Contrary situations in which $\Sigma_t \to \infty$ call for *no pooling*: there is no relationship between adjacent parameter estimates as in Gürkaynak et al. (2007). Finding the right degree of pooling over time is critical for getting sensible yield curve estimates from our historical data. In appendix B, we show that the no-pooling case significantly restricts the number of periods for which we can get estimates and leads to highly volatile yield estimates.

As is standard, we assume that the gold dollar discount function $\mathsf{q}_t$ can price coupon-bearing government gold bonds well.

**Assumption 2.** The law of one price holds for government bonds and for each $t \geq 0$.

[6] In our context, "stochastic volatility" means that the amount of pooling varies over time. We gain additional pooling by letting shocks to different components of $\lambda_t$ be correlated. Assuming that different parts of the yield curve follow correlated but time-invariant dynamics allows us to transmit what we learn about co-movements between short- and long-term yields from years when many maturities are outstanding to years when data are scarce.

That is, the price of bond $i$ with promised payments $\mathsf{m}_t^{(i)}$ is given by:

$$p_t^{(i)} = \sum_{j=1}^{\infty} q_t^{(j)} m_{t+j}^{(i)}.$$

Assumption 2 expresses our key identifying restriction: within each time period, a common discount function prices all bonds that we include in our sample, i.e., there is no cross-sectional variation in how government promises of bond repayments are priced. In principle, $\mathsf{q}_t$ implicitly includes compensations for default risks, convenience benefits, and inflation risk, so it should be thought of as the price of a *risky* government promise. Our specification allows these components to vary with maturity $j$ and time $t$, but not by individual bond. However, as we noted in Section 2, 19th century US federal bonds had idiosyncratic contract features that could easily lead to violations of this common discount function assumption. For this reason, we introduce *bond specific* pricing errors.

**Assumption 3.** The observed price of bond $i$ with promised payments $\mathsf{m}_t^{(i)}$ differs from the price in Assumption 2 by an independent pricing error that has a time-invariant Gaussian distribution with mean 0 and standard deviation $\sigma_m^{(i)}$. That is, we have:

$$\tilde{p}_t^{(i)} = \sum_{j=1}^{\infty} q_t^{(j)} m_{t+j}^{(i)} + d_t^{(i)} \sigma_m^{(i)} \varepsilon_t^{(i)} \tag{3.3}$$

where $\tilde{p}_t^{(i)}$ denotes the *observed* period-$t$ price of bond $i$ in terms of gold and $d_t^{(i)}$ is the Macaulay duration of bond $i$ in period $t$.[7]

Any yield curve model with fewer free parameters than number of bonds to be fit must allow for pricing errors. What gives content to yield curve models is the choice of the dimension in which the pricing error can and cannot vary, since this choice identifies the type of data variation that the model utilizes during estimation.[8] In our setup there are three candidate dimensions in which we could let the pricing error vary: bond, time, and maturity. In a modern context, homogenization of the sample through a careful pre-selection of bonds is often feasible, which makes bond-specific pricing errors rather wasteful and

[7]We adjust the price error by the inverse duration to ensure that the error in yield terms does not become unbounded as the bond nears maturity. To a first order, the difference between actual and predicted prices of bond $i$ equals its duration multiplied by the difference between actual and predicted yields. As a result, minimizing inverse-duration-weighted pricing errors approximates minimizing unweighted yield errors.

[8]Naturally, when each observation has a separate pricing error the model parameters are not identified. Making the pricing error fixed for a particular subset of observations allows the model to detect and automatically down-weight data points that appear "outliers".

time-dependent pricing errors—as in the period-by-period estimation of Gürkaynak et al. (2007)—more appropriate. The sparse nature of historical data sets forces us to economize on the number of bonds that we include in our sample. The key insight behind Assumption 3 is that a set of bonds with heterogeneous contract features can still provide valuable information on a common yield curve, but to extract this information we need to account for their heterogeneity during the estimation. Our bond-specific pricing errors do exactly that by decreasing the influence of peculiar bonds on our yield estimates.

Nonetheless, we find that short-term Treasury notes have particularly large estimated bond specific pricing errors.[9] We suspect the large relative pricing errors reflect a liquidity premium that emerges from the relative ease in which such bonds could be used for transactions. Ultimately, this led us to exclude *all* bonds with less than one year to maturity from our sample. We reconsider the pricing of short-term bonds in Subsection 7.1 when we discuss the so called "short-rate-disconnect".

Assumptions 1-3 give rise to a non-linear state space model with state equations (3.1)-(3.2) and observation equation (3.3). We estimate this model with Bayesian Markov Chain Monte Carlo (MCMC) methods assuming the weakly informative priors detailed in Table I. In Appendix B we provide further details on the estimation. For more discussions on how our modelling choices tackle some of the challenges posed by the historical sample see our companion paper Payne et al. (2023a).

Remarks: In a companion paper, Payne et al. (2023a), we conduct an information-criterion-based model selection exercise and find that a model with stochastic volatility and correlated shocks is preferred to a model without these components because of how these features help reduce pricing errors without overfitting. We also show that introducing mean-reversion in the factor process does not significantly improve the fit, so we omit mean reversion from our baseline model. Our approach outlined in this section estimates parameters directly from bond prices adjusted by the bond's duration using the same approach as Gürkaynak et al. (2007). An alternative approach in the literature is first to produce zero-coupon yield observations by an iterative extraction method—the so called "bootstrap" of Fama and Bliss (1987)—and fit a DNS model to the resulting yield series. With our sparse data set, properties of the Fama-Bliss "bootstrap" are delicate, so we prefer instead to minimize inverse-duration-adjusted price errors.

[9] These are the Treasury notes that the US issued during the War of 1812 and the Mexican-American War along with the various issues of the Treasury Note of 1860.

Table I: Prior Distributions of the Dynamic Nelson-Siegel Model

| Parameter | Description | Priors | Hyper parameters |
|---|---|---|---|
| $L_0$ | initial level factor | $\mathcal{N}(\mu, \sigma^2)$ | $(\mu, \sigma) = (10, 5)$ |
| $S_0$ | initial slope factor | $\mathcal{N}(\mu, \sigma^2)$ | $(\mu, \sigma) = (0, 10)$ |
| $C_0$ | initial curvature factor | $\mathcal{N}(\mu, \sigma^2)$ | $(\mu, \sigma) = (0, 10)$ |
| $\tau$ | location parameter | $\log \mathcal{N}(\mu, \sigma^2)$ | $(\mu, \sigma) = (5, 5)$ |
| $\sigma_0$ | initial shock volatilities | $\log \mathcal{N}(\mu, \sigma^2)$ | $(\mu, \sigma) = (0.05, 0.1)$ |
| $\Omega$ | correlation of shocks | LKJ$(\eta)$ | $\eta = 5$ |
| $\Xi_\sigma$ | volatilities of $\sigma_t$ shocks | Exp$(\lambda)$ | $\lambda = 20$ |
| $\sigma_m^{(i)}$ | volatilities of pricing error | Exp$(\lambda)$ | $\lambda = 0.1$ |

[1] The LKJ distribution is defined by $p(\Omega|\eta) \propto \det(\Omega)^{\eta-1}$. See Lewandowski et al. (2009)

## 3.2 Fit

An important aspect of our approach is our use of bond-specific pricing errors. The black crosses in Panel A of Figure II depict *inverse-duration-weighted mean absolute pricing errors* for each bond included in the analysis.[10] We note that our yield curves price the included bonds well with similar errors across different bonds. This is also reflected in estimated standard deviations of bond-specific pricing errors, $\sigma_m^{(i)}$, the posterior distributions of which are depicted by the boxplots in Panel A of Figure II. The relative magnitude of these estimates indicates the influences of particular bonds on the estimated yield curves. Our algorithm assigns relatively less weight to bonds with large estimated $\sigma_m^{(i)}$ values. The absence of obvious "clusters" in Panel A indicates that our common discount function assumption provides a reasonably good description of the gold dollar bonds with maturities larger than 1 year.

To obtain a measure of fit with a more interpretable scale, we take the posterior median of our zero-coupon yield estimates, compute the implied yields-to-maturity for each bond at each month, and compare them to the observed yields-to-maturity. The bottom panels of Figure II report different aspects of these *yield errors*. Panel B depicts distributions of yield errors for specific maturity bins. On average our parametric yield curve specification fits observed yield-to-maturities well and without systematic differences across maturities larger than 1 year. Panel C depicts cross-sectional means and standard deviations (over

[10] Bliss (1996) used this measure of in-sample fit. It is computed as the time average of the absolute difference between observed prices and posterior median price forecasts weighted by the inverse of the respective Macaulay durations.

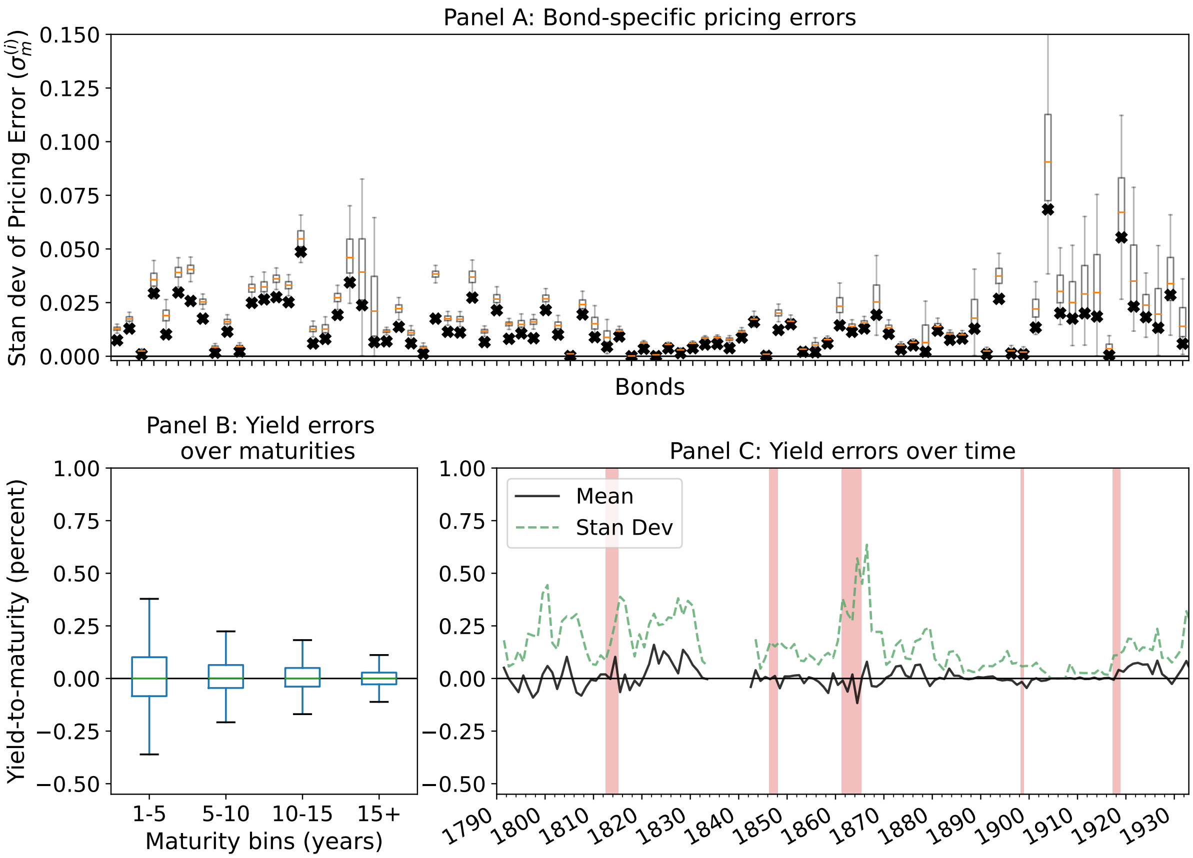


Figure II: Statistical Fit of the Model

*Panel A:* Each boxplot represents the interquartile range and median (orange line) of the posterior distribution of $\sigma_m^{(i)}$—that is the standard deviation of the bond-specific pricing error—for each bond used in the estimation. Black crosses represent *duration-weighted mean absolute price errors* computed from the difference between observed and model-implied prices for each bond. *Panel B:* Each boxplot represents the interquartile range and median (green line) of yield errors (difference between model-implied and observed yield-to-maturities) for different maturity bins. *Panel C:* Cross-sectional mean (black solid line) and standard deviation (green dashed line) of yield errors over time. Sample statistics are computed over bonds for each calendar year. Red bars correspond to wars.

bonds for each calendar year) of yield errors. The mean error stays close to zero within a similar range to Diebold and Li (2006).[11] The cross-sectional variation in the yield error is also typically small but becomes large relative to Diebold and Li (2006) during the early 19th century and the Civil War, indicating that we have the most difficulty pricing the cross section of bonds during those years. This means that our yield curve estimates should be regarded as more tenuous both during big wars, when yields are rapidly changing, and during the early 19th century, when we have limited data.

# 4 Gold Yield Curves: 1791-1933

In this section, we use our estimates of gold denominated US federal yield curves to infer that: (i) yields trended downwards throughout the 19th century, (ii) there were larger spikes in yields during 19th century wars than has previously been understood, and (iii) slopes of yield curves were typically negative before the Civil War, but positive afterwards.

## 4.1 Long term US yields fell during the 19th century

Figure III depicts selected long term yield estimates. The solid black and grey lines depict the median of our posterior probability distributions for 10-year, gold denominated, zero coupon yields. Bands around the posterior median depict the 90% interquantile range. The estimates before the War of 1812, and especially before 1800, have wider interquantile ranges and so should be treated with caution. This reflects that few bonds close to maturity traded during those times. For this reason, we show the 25 year gold dollar zero-coupon yields for dates before 1800.

Long yields trended downward throughout the 19th century interrupted by sharp (but temporary) increases during times of war and financial turmoil. The sharpest increases came during the War of 1812 and the Civil War.

Few estimates of 19th century US long yields exist. Most economic historians follow Officer and Williamson (2021) and use a 'composite series' that combines the Homer and Sylla (2004) estimates for 10-year US federal bond yield for the period from 1798-1861 with the yield-to-maturity on New England Municipal bonds for 1862-1899 and the yield-to-maturity on corporate bonds for 1900-1940.[12] Figure III also plots this composite series

[11] For comparison, in Table 2, they show time averages of yield errors across maturities in the range of -6 to +6 basis points. In view of our much more limited data, we find this encouraging.

[12] Prominent examples of researchers who used this series include Shiller (2015), Jordà et al. (2019), and Hamilton et al. (2016), as well as the Macrohistory Database (Jordà et al., 2016).

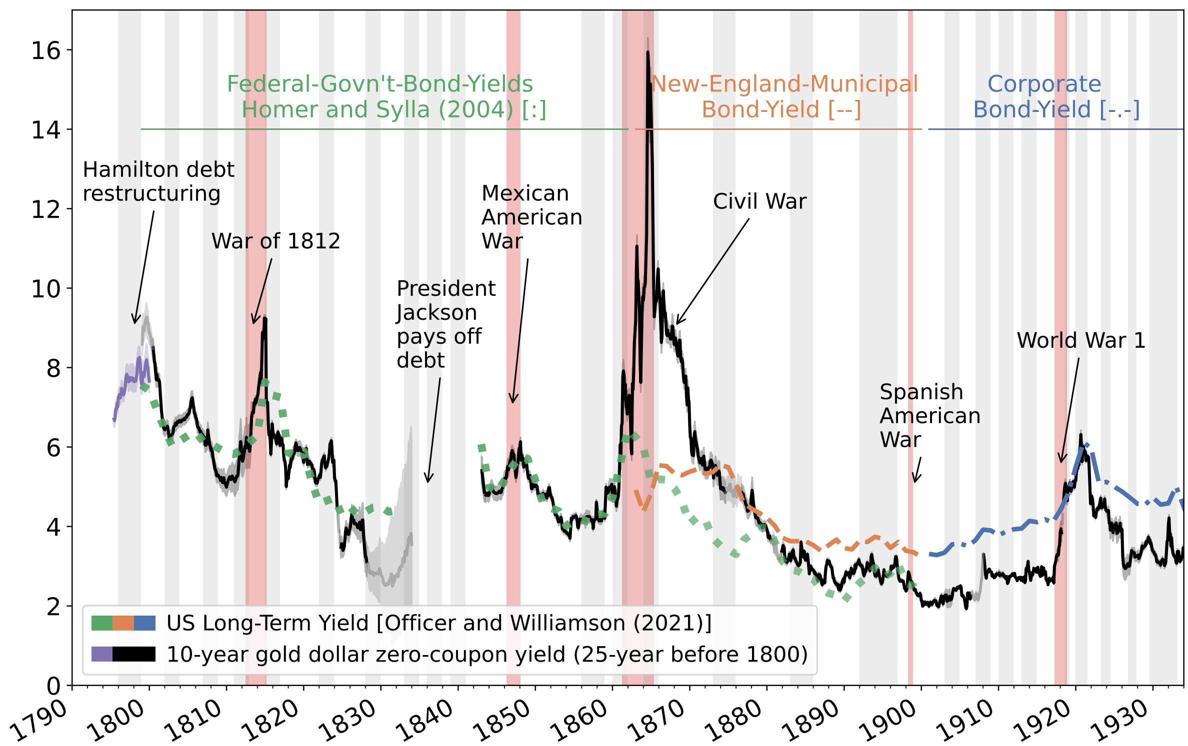


Figure III: Long-Term Yield Estimates

The solid black (purple) line depicts the median of our posterior estimate for the 10-year (25-year), gold denominated, zero coupon yield after (before) 1800. The bands around the posterior median depict the 90% interquantile range. We use a grey line for periods when the 10-year maturity was not covered by observed maturities and so the estimate should be thought of as an extrapolation. The green line (dotted) depicts the 'US Government Bond Yield' series from Homer and Sylla (2004). The orange line (dashed) depicts the New England Municipal Bond Yield reported by Homer and Sylla (2004). The blue line (dash-dotted) depicts the Corporate Bond Yield reported by Homer and Sylla (2004). The combined green-orange-blue line depicts the 'composite' bond series used by Officer and Williamson (2021). The light gray intervals depict recessions and the light red intervals depict wars.

alongside our 10-year yields. Evidently, compared to the composite series, our estimates are similar before the Civil War, much higher during the Civil War, and consistently lower after greenback-gold convertibility in 1879. In particular, our ten-year gold yield estimate reaches a peak of 16% near the end of the Civil War, which is substantially higher than the Homer and Sylla (2004) series peak of 6% at the start of the war.

We find it reassuring that our estimates align with Homer and Sylla (2004) during "non-emergency" periods because there are good reasons to think that their estimates should be a good approximation to the 10 year yield.[13] The following observations also suggest that during the Civil War our higher estimates are more plausible than those of Homer and Sylla (2004).[14] Starting in 1862, all US Treasury bonds could be purchased with greenback dollars, including bonds with coupons and principal payments denominated in diverse units of account, some in greenbacks, others in gold dollars. The value of the greenback fluctuated with battlefield and political news, and all Treasury bond prices deviated substantially from par. For example, with the re-election of President Abraham Lincoln in doubt during the summer of 1864, 100 greenback dollars could be purchased for as few as 40 gold dollars. Consequently, during that time Treasury bonds that promised to pay 6 percent coupons in gold dollars could be purchased for 40 percent of par, implying long-term yields in excess of 15 percent.

Our estimates of lower long-term yields after 1880 reflect differences between US federal debt and other bonds during the late 19th century. Federal debt probably carried greater default risks during the Civil War. After the war, federal debt played a special role in the financial sector because National Banking Era protocols increased demands for federal bonds as reserves against National Bank Notes. In particular, we can see that a "convenience yield" spread opened up between US federal bonds and other low risk bonds during the 1880s that persists to this day. Because it mixes very different types of bonds, the commonly used composite long-term yield series inaccurately reflects variations in either US federal borrowing costs or private sector borrowing costs.[15]

[13]Their approach calculates an average yield to maturity for 10 year bonds trading close to par, which should be similar to the 10 year zero-coupon yield when the yield curve is relatively flat. Except during and after the Civil War, the average duration of outstanding bonds was close to 10 years and the average market trading price was close to par so Homer and Sylla (2004) have enough bonds for their methodology.

[14]Homer and Sylla (2004) warn against using their estimates for the Civil War period stating on page 303, "...the tables of bond yields for the years 1863 to 1870 do not provide a reliable picture of long-term interest rates." This is because there were no federal bonds trading with a gold price of par, so they were forced to estimate the yield as the gold coupon rate for bonds trading with a greenback price of par. We capture greater variation in the yield curve because we use the universe of US Treasury bonds at monthly frequency whereas Homer and Sylla (2004) use the subset of these bonds that traded at par.

[15]Comparing our yield estimates to those of Homer and Sylla (2004) sheds new light on a literature that, starting with Evans (1985, 1987), found that during the 19th century there was no strong association

## 4.2 Slopes switched signs

There are no existing estimates for 19th century yields on short term US federal debts. Instead, 19th century economic historians have typically presented the bank call rate as a short yield.[16] Figure IV depicts our estimates for 2-year gold denominated zero-coupon yields alongside the bank call rate. Our short term yield series is substantially higher than the bank call rate during the Civil War and does not always co-move with it after the Civil War.[17] This is not surprising because the bank call rate reflects distress in the financial sector whereas the 2-year zero coupon yield on US federal debt reflects short term federal government borrowing costs. Evidently the federal government had substantially more difficulty borrowing during the war but less difficulty borrowing during financial crises when the call rate spiked.

Having yield curves allows us to study their slopes. Figure V depicts the 10-year gold dollar yield minus the 2-year yield, which we refer to as the term spread. A positive term spread indicates an upward sloping yield curve (i.e., longer maturity bonds have higher rates), while a negative term spread indicates an inverted yield curve (i.e., shorter maturity bonds have higher rates). The term spread was typically negative before the Civil War and slightly positive afterwards, with major decreases occurring during the War of 1812, the Mexican-American War, and the Civil War.[18] This differs from the post-WWII period when the yield curve persistently slopes upwards (Gürkaynak and Wright, 2012).

Altered inflation dynamics around the Civil War is a possible explanation the sign switch in the slope of the yield curve. Our analysis of historical inflation in Payne et al. (2023b) suggests that long term inflation was relatively hard to predict before the Civil War but relatively easy to predict after the Civil War. This change coincides with the sign switch of the slope of the yield curve from negative to slightly positive, which suggests that the term spread becoming positive might be connected to the decrease in the long

between interest costs and deficits. To infer that, previous papers used the composite series in Figures III and IV. Our estimates indicate that those series substantially underestimated increases in yields on US federal debt during episodes of large 19th century government deficits. However, we caution that we are not able to make any claims about the causal connection between deficits and interest costs in this paper.

[16] For example, Officer and Williamson (2021) and Jordà et al. (2016) use a series "Surplus Funds (Contemporary Series)." The Series is the interest rate on the short-term lending or borrowing of surplus funds by financial institutions, often referred to as the "call rate".

[17] Anecdotal evidence indicates that Union short-term debt paid very high yields during the Civil War. For example, Homer and Sylla (2004, page 302) report that in 1860 the Treasury had issued one-year notes at rates of 10-12% and had rejected bids ranging from 15-36%.

[18] These observations raise interesting questions about the connection between the issuance patterns in Figure I and the slope of the yield curve. Addressing causality could be an interesting area for future study but would require an instrument for sovereign debt issuance.

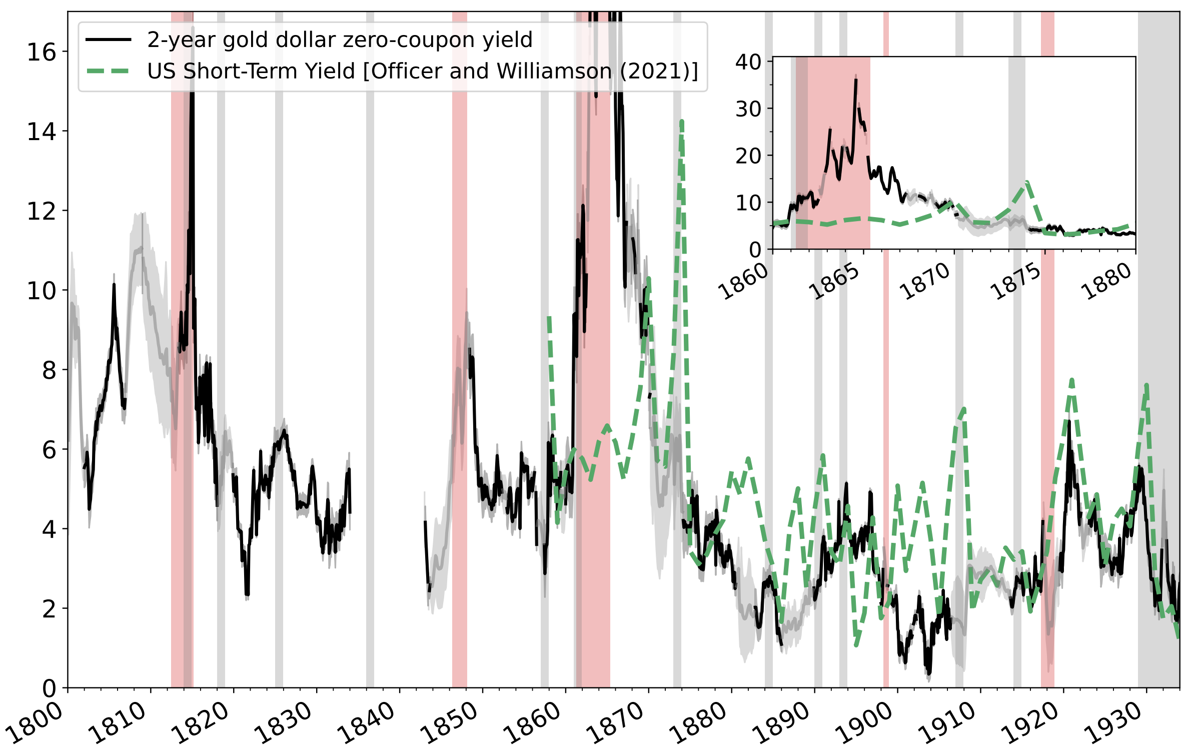


Figure IV: Short-Term Yields

The solid black line depicts the median of our posterior estimate for the 2-year, gold denominated, zero coupon yield. The grey bands around the posterior median depict the 90% interquantile range. We use a grey line for periods when the 2-year maturity was not covered by observed maturities and so the estimate should be thought of as an extrapolation. The green dashed line depicts the US short term yield series (surplus funds, contemporary) used by Officer and Williamson (2021) and Jordà et al. (2019). The gray intervals depict financial crises from Reinhart and Rogoff (2009). The light red intervals depict wars (from left to right: the War of 1812, the Mexican-American War, the Civil War, the Spanish-American War, and World War I).

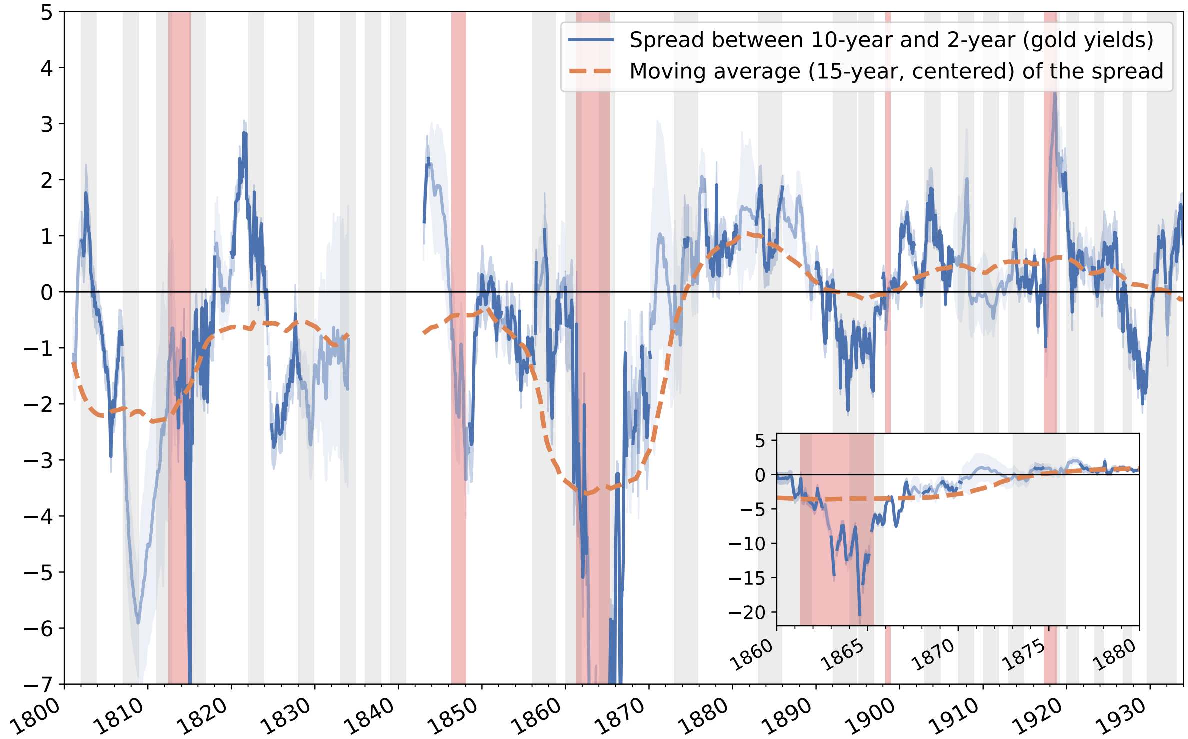


Figure V: Term Spread

The solid blue line depicts the posterior median for the spread between the yield on 10-year, gold denominated, zero coupon US government bonds and the yield on 2-year bonds. The grey bands around the posterior median depict the 90% interquantile range. We use a grey line for periods when either the 10 or the 2 year maturity was not covered by observed maturities and so the estimate should be thought of as an extrapolation. The orange dashed line depicts the 15-year centered moving average of the spread.

run "inflation risk." This relationship would be consistent with asset pricing theory if the inflation risk premium was negative, i.e., if federal gold bonds provided good hedges against inflation. Payne et al. (2023b) offer evidence that this was indeed the case in the 19th century.

# 5 Long Term US Yields Approached UK Yields

One possible explanation for the sustained decrease in long yields is a fall in the risk premium on US federal debt securities. After the American War for Independence, the Continental Congress owed approximately $52 million in foreign loans to France, Spain, and Holland, loan office and debt certificates to the American public, and unpaid interest. The Congress confronted substantially higher long term yields than the UK even though the UK then had a high debt-to-GDP ratio. This situation sparked a lively debate in the US about whether and how to service wartime debts. In his *1790 Report on Public Credit*, Treasury Secretary Alexander Hamilton argued

> For when the credit of a country is in any degree questionable, it never fails to give an extravagant premium, in one shape or another, upon all the loans it has occasion to make. Nor does the evil end here; the same disadvantage must be sustained upon whatever is to be bought on terms of future payment.

Hamilton convinced Congress to honor its foreign debts and to offer its domestic creditors a rescheduling deal that, except for owners of Continental and state-issued currencies, left them with 65 to 80 cents on the dollar. Even after those haircuts, US domestic creditors came away with much higher *ex post* returns than had been anticipated during much of the previous decade.[19] Hamilton intended these steps to kick-start a process that would eventually give the US a reputation for servicing its debts that would help reduce future US borrowing costs to lower levels than being paid by the UK government.

We use our estimated yield curves on US federal debt to quantify whether and when Hamilton's hopes were fulfilled. Figure VI compares yields-to-maturity on gold denominated UK consols to yields-to-maturity on *hypothetical* gold denominated US consols that promise the same coupon flows as the UK consols.[20] We plot yields to maturity on gold

[19] Hall and Sargent (2014) describe details of Hamilton's rescheduling operations, including estimates of haircuts to domestic federal creditors.

[20] The UK consol yield is the series "Spliced consol yield 1753-2015, corrected for Goschen's conversion issues" from Thomas and Dimsdale (2017). The hypothetical, gold denominated US consols promise the same annuity coupon payments as those in the UK consol yield series.

denominated UK consols because almost all UK government bonds were consols, so that is the only UK yield that can be reliably estimated. Our estimates of yield curves enables the construction of yields on comparable hypothetical US consols. However, when constructing the synthetic US series we are mindful of the Nelson-Siegel model's unreliable extrapolation properties.[21] As a result, when it comes to extrapolation, we use flat rate extrapolation instead of our estimated Nelson-Siegel yield curves.[22]

We offers the following observations. First, notice that the hypothetical US consol yield exhibits a downward trend, falling from close to 8% at the beginning of the 19th century to around 2% at the end of the century. Second, notice that the US yield was typically higher than the UK yield until the 1880s, except for a temporary coincidence during the 1820s. Third, notice that the US yield was persistently lower than the UK yield after 1900. This suggests that the combination of the federal government's having serviced War of Independence IOUs, albeit with substantial haircuts to domestic creditors, and having completely retired all federal debt by the mid-1830s, along with activities of the First and Second Banks of the United States had made significant contributions to Hamilton's project. However, the reemergence of the spread between US and UK debt during the period from 1840-1880 suggests that Hamilton's plan was not fully realized until disruptions from the Civil War had passed, the National Banking system had been established, and gold-greenback parity had been set in place in 1879. Evidently, the process of reordering US financial arrangements heavily influenced prices of federal bonds in the 1860-70s.

Differences between the yields to maturity on the UK consol and on the hypothetical US consol probably reflect different default risks. UK debt was considered a "safe-asset" during the 19th century, whereas US wars and state-level defaults induced investors to regard 19th century US debt as risky. If there was no default risk on UK consols and gold inflation expectations were similar in the US and UK, then we can interpret the difference between US and UK consol yields as reflecting a risk premium on US federal debt. Under this interpretation, Figure VI indicates that US federal debt carried a risk premium until the late 19th century when it became another "safe-asset." Our estimates show that this

[21] Gürkaynak et al. (2007) describe why the Nelson-Siegel model can lead to unreliable inferences on very long maturity yields especially in the presence of the so called "convexity effect". They strongly advise against using the Nelson-Siegel model for extrapolation. Indeed, while our estimated level factor, $L_t$, appears reasonable for most of our sample period, we find that during the early 1880s and around the turn of the 20th century the estimated $L_t$ becomes slightly negative suggesting, that the convexity effect was strong during these subperiods.

[22] More precisely, let $j_t^{\max}$ denote the months-to-maturity of the bond that has the longest maturity in period $t$. For each $t$ we use a composite yield curve which coincides with our estimated Nelson-Siegel yield curve for $j \leq j_t^{\max}$ and is given by $y_t^{(j)} = y_t^{(j_t^{\max})}$ for all $j > j_t^{\max}$.

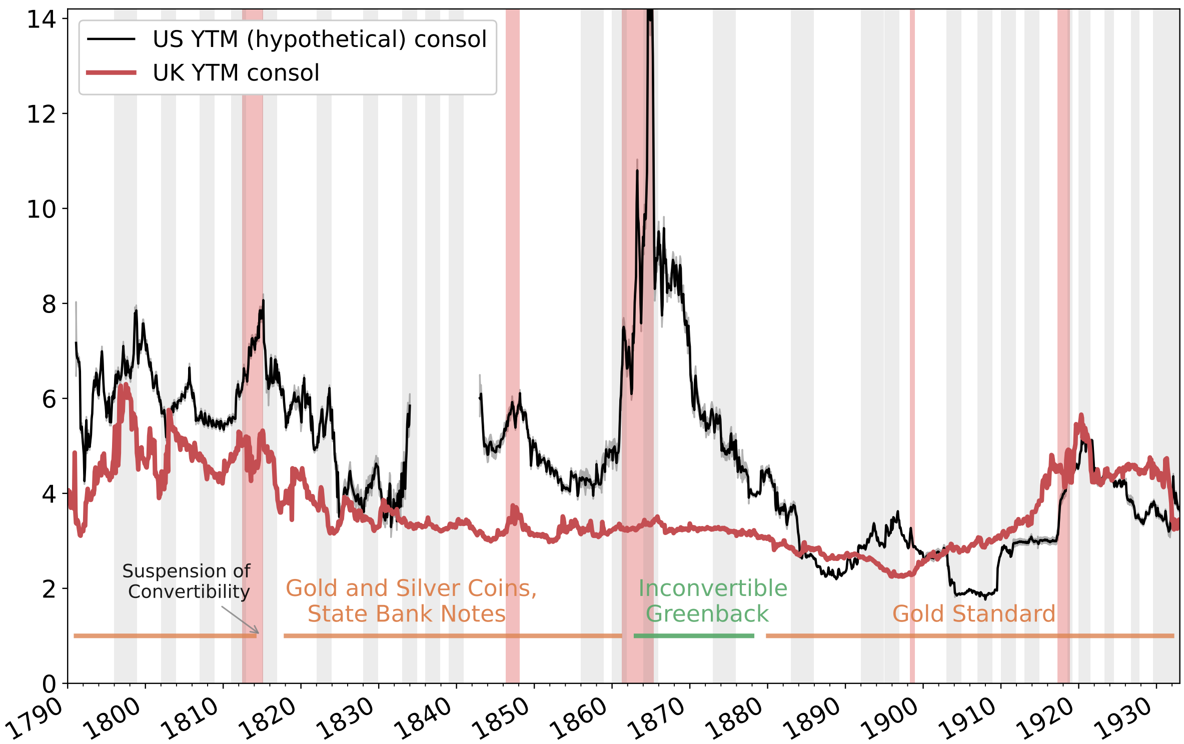


Figure VI: US and UK Consol Yields.

The solid black line depicts the median of our posterior estimate for the yield-to-maturity on hypothetical gold denominated US consols that promise the same coupon flows as the UK consols. The grey bands around the posterior median depict the 90% interquantile range. The red line depicts the UK long-term yield (implied by the 3% consol price) from Thomas and Dimsdale (2017). The light gray intervals depict recessions, and the light red intervals depict wars.

occurred well before US debt became the world's main reserve asset after World War II. Evaporation of those risk-premia thus set the stage for a realignment of global finance that ultimately let US government debt replace UK debt as the global "safe-asset" during and after the 1944-1971 Bretton Woods years.

# 6 Greenback Era: 1862-1878

How were government borrowing costs affected by monetary policy disruptions during 1862-1878, when greenback dollars traded at volatile discounts relative to gold dollars? Figure VII shows greenback-to-gold dollar exchange rates as well as prices of bonds that promised payments in gold. Evidently, bond prices started to track the greenback-to-gold exchange rate only for bonds nearing maturity. This suggests that investors did not expect a persistent deviation from greenback-to-gold parity, despite substantial devaluations of greenback dollars during the Civil War, indicating the weight of of a heavy nominal anchor. We shall formalize this idea by estimating expectations about future exchange rates and a greenback denominated yield curve. Our statistical challenge is that we have very few greenback bonds and most of them are short term. This means that we cannot estimate the greenback yield curves purely using price data for greenback paying bonds. Instead, we use an approach that also incorporates our gold yield curves and data on the gold to greenback exchange.

## 6.1 Historical Context: Greenback Dollars

Before the Civil War, the US dollar was defined in units of gold and silver. The federal government minted gold and silver coins and did not issue paper dollars. Under financial strain from the Civil War, on February 25, 1862, Congress passed a Legal Tender Act that authorized the Treasury to issue 150 million dollars of a paper currency known as greenbacks that the government did not promise to exchange for gold dollars on demand. Subsequent acts authorized the Treasury to issue more notes, eventually totaling 450 million dollars. Investors could use greenbacks to purchase bonds from the federal government at their par values. Gold dollars continued to be used to settle international transactions, to pay US tariffs, and to pay coupons on some federal bonds. Figure VII shows that from 1862 to December 31, 1878 greenback dollars traded at discounts relative to gold dollars. On January 1, 1879, the US Treasury started converting greenbacks into gold dollars one-for-one.

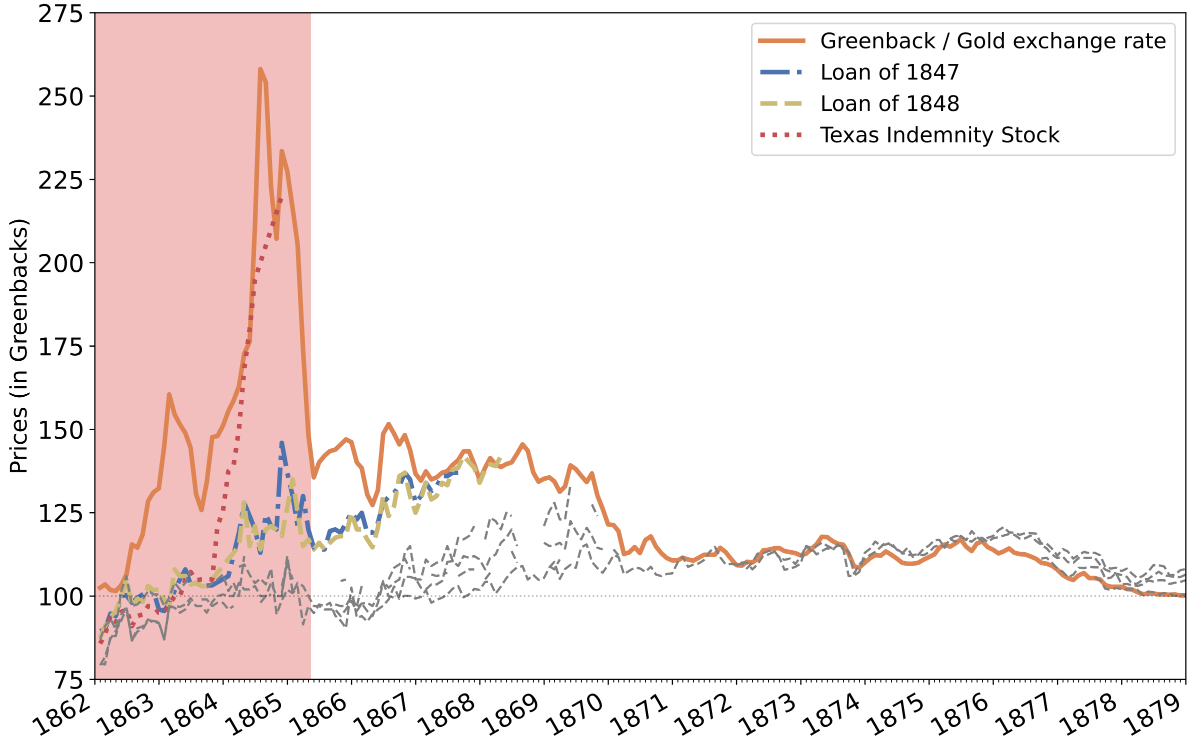


Figure VII: Prices of Gold and Bonds: 1860-1880.

The solid orange line depicts the greenback-to-gold exchange rate (expressed as the number of greenback dollars required to purchase 100 gold dollars). The colored dotted, dashed, and dash-dotted lines depict observed prices (denominated in greenbacks) for some outstanding bonds maturing before 1868. The grey lines depict prices of bonds that matured after 1868. The light red interval depicts the Civil War.

## 6.2 Estimation Strategy

Our estimation strategy incorporates our gold yield curves, price data for greenback paying bonds, and data on the gold to greenback and gold to goods exchange rates. We start by positing a form of interest rate parity in the sense that we attribute any difference between gold and greenback denominated yields (with matching maturity) to the market's expectations about future movements in the gold-to-greenback dollar exchange rate.

**Assumption 4.** The gold price of a promise to one *greenback dollar* at time $t+j$ is:

$$q_t^{(j,g)} = q_t^{(j)} z_t^{(j)} \qquad \text{with} \quad z_t^{(j)} := \mathbb{E}_t[P_{t+j}]/P_t$$

where $P_t$ is the gold-to-greenback exchange rate in period $t$, $z_t^{(j)}$ is a "multiplier" that converts $(t+j)$-period greenback dollars to $(t+j)$-period gold dollars, and $q_t^{(j)}$ is the gold price of a government promise to one gold dollar at time $(t+j)$.

To approximate the market's exchange rate forecasts, $\mathbb{E}_t[P_{t+j}]$ for all $j \geq 1$, we use a flexible vector autoregressive (VAR) model that summarizes the dynamic relationship between two key exchange rates: the gold-to-greenback dollar exchange rate, $P_t$, and the gold-to-goods exchange rate, $G_t$. To avoid enforcing a fixed long-run mean ("nominal anchor") and a fixed rate of convergence to this mean, we allow the unconditional mean and autoregression matrix of the VAR to vary over time.

**Assumption 5.** The vector of exchange rates $v_t := \{P_t, G_t\}$ follows a vector autoregressive model with time-varying parameters:

$$v_{t+1} = \mu_t + A_{1,t}\left(v_t - \mu_t\right) + A_{2,t}\left(v_{t-1} - \mu_t\right) + \varepsilon_{v,t+1} \qquad \varepsilon_{v,t+1} \sim \mathcal{N}\left(\mathbf{0}, \Sigma_v\right) \tag{6.1}$$

where $\mu_t$, $A_{1,t}$ and $A_{2,t}$ follow random walks with independent Gaussian shocks.[23]

Assumption 5 asserts that $P_{t+j}$ is conditionally independent of the pricing kernel implicitly contained in $q_t^{(j)}$ and that all relevant information for forecasting $P_{t+j}$ is contained in observable prices. Since the principal source of variations in the gold-to-greenback dollar exchange rate was most likely "default risk" due to battlefield losses and political sentiments, this assumption can be interpreted as saying that all information relevant

[23] We conducted an information-criterium-based model selection exercise to see which *fixed parameter* model from the VARMA(p, q) class can describe the dynamic relationship between $P_t$ and $G_t$ best for the period 1862-1879 and found that it is very difficult to differentiate between the VAR(2), VAR(3), and VARMA(1, 1) models. We chose the VAR(2) model to keep the analysis as tractable as possible. We thank anonymous Referee 1 for suggestions about modelling the exchange rate process.

for forecasting the financial effects of the war is contained in the gold-to-greenback dollar exchange rate and the price level. We find it reassuring that the greenback-to-goods exchange rate tracks the gold yield curve during big Civil War changes.

We use the observed exchange rates, $v_t$, to estimate the parameters of the time-varying VAR in (6.1). In addition, however, we also want to use the relative price differentials between the available greenback and gold dollar bonds—accompanied with the interest rate parity relation from Assumption 4—to inform our estimation of the exchange rate model and make sure that the VAR-implied exchange rate forecasts are consistent with both the empirical distribution of $v_t$ and the pricing of government bonds. To that end, when we estimate the time-varying VAR, we require that the model-implied exchange rate forecasts are consistent with the price of greenback dollar bonds according to:

$$\tilde{p}_t^{(i,g)} = \sum_{j=1}^{\infty} q_t^{(j)} z_t^{(j)} m_{t+j}^{(i,g)} + d_t^{(i,g)} \sigma_m^{(i)} \varepsilon_t^{(i)}, \tag{6.2}$$

where $\tilde{p}_t^{(i,g)}$ denotes the *observed* period-t price of greenback dollar bond $i$ in terms of gold dollar, $\{m_{t+j}^{(i,g)}\}_{j\geq 1}$ is the corresponding sequence of greenback dollar coupon and principal payments, and the "multiplier" $z_t^{(j)}$ is determined by the time-varying VAR in line with Assumption 5. As in Section 3, we use bond-specific pricing errors, $\sigma_m^{(i)}$, scaled by the Macaulay duration $d_t^{(i,g)}$ of bond $i$.

We estimate the exchange rate model *conditional on the gold yield curve*—summarized by $\mathsf{q}_t$ in formula (6.2). Because we wish to incorporate our uncertainty about $\mathsf{q}_t$, we specify a prior distribution for $\mathsf{q}_t$. In particular, we assume that the gold yield curve parameters, $\{\lambda_t\}$ and $\tau$, follow a multivariate normal distribution and set the corresponding mean and covariance matrix equal to their posterior sample analogues from Section 3.[24] Together with priors for the remaining parameters detailed in Table II, we estimate the model (6.1)-(6.2) with Bayesian MCMC techniques. In Appendix B we provide further details on the estimation.

Remarks: Figure I indicates that there are few greenback dollar bonds in our sample and that most of them are short-term. This means that we cannot reliably fit a DNS model just using the greenback bonds. Instead, we model the greenback yield curve as a *combination* of two objects aboout which we can make plausible inference despite data

[24]We view this specification of the prior for $\mathsf{q}_t$ as a way to "condition" on our baseline gold yield curve from Section 3, since we expect the posterior for $\mathsf{q}_t$ to move very little relative to the corresponding prior. We verify that this is indeed the case.

Table II: Prior Distributions of the Exchange Rate Model

| Parameter | Description | Priors | Hyper parameters |
|---|---|---|---|
| $\mu_0$ | initial long run mean | $\mathcal{N}(\mu, \sigma^2\mathbb{I}_2)$ | $\mu = [1, 1.2]'$, $\sigma = 1$ |
| $A_{1,0}$ | initial persistence matrix | $\mathcal{N}(\mu, \sigma^2\mathbb{I}_4)$ | $\mu = [0, 0, 0, 0]'$, $\sigma = 1$ |
| $A_{2,0}$ | initial persistence matrix | $\mathcal{N}(\mu, \sigma^2\mathbb{I}_4)$ | $\mu = [0, 0, 0, 0]'$, $\sigma = 1$ |
| $\Omega_v$ | forecast error correlations | LKJ$(\eta)$ | $\eta = 2$ |
| $\Xi_v$ | forecast error volatilities | Exp$(\lambda)$ | $\lambda = 20$ |
| $\Xi_\mu$ | volatilities of $\mu_t$ shocks | Exp$(\lambda)$ | $\lambda = 50$ |
| $\Xi_A$ | volatilities of $A_t$ shocks | Exp$(\lambda)$ | $\lambda = 50$ |
| $\sigma_m^{(i)}$ | volatilities of pricing error | Exp$(\lambda)$ | $\lambda = 0.1$ |
| $(\{\lambda_t\}, \tau)$ | parameters of $\{\mathsf{q}_t\}$ | $\mathcal{N}(\mu, \Sigma)$ | from Section 3 posterior |

[1] We decompose the forecast error covariance as $\Sigma_v = \Xi_v \Omega_v \Xi_v$.

limitations: a Nelson-Siegel gold dollar yield curve, and a flexible model of exchange rates. As we show in Subsection 6.3, for most of the Greenback Era estimated greenback yields are very different from their gold dollar counterparts, suggesting that our exchange rate model adds significant flexibility relative to the gold dollar yield curve. Importantly, to inform our exchange rate model we include not only the exchange rates but also the *relative* prices of greenback and gold denominated government bonds.[25] We believe that this approach is a fruitful way to utilize the information in our sample. Nevertheless, our estimates of the greenback dollar yields should be treated with more caution than the gold dollar yields in Section 3.

## 6.3 Revisiting a Civil War Yield Puzzle

Figure VIII shows our estimates of long term greenback denominated yields from 1862-1878. We can observe that the interquartile ranges are higher for the greenback yields and so these estimates should be treated with more caution. As already discussed, this reflects that there were relatively few bonds paying in greenbacks and so we have to rely on Assumptions 4, 5, and other data sources.

The greenback yield is systematically below the gold denominated yield. In particular,

[25] The observed price differentials between greenback and gold denominated bonds of matching maturities turn out to be critical for the estimated greenback dollar yield curve. We find that estimating the time-varying VAR with restriction (6.2) implies a more pronounced time-variation in $\mu_t$, $A_{1,t}$ and $A_{2,t}$ than a model without restriction (6.2).

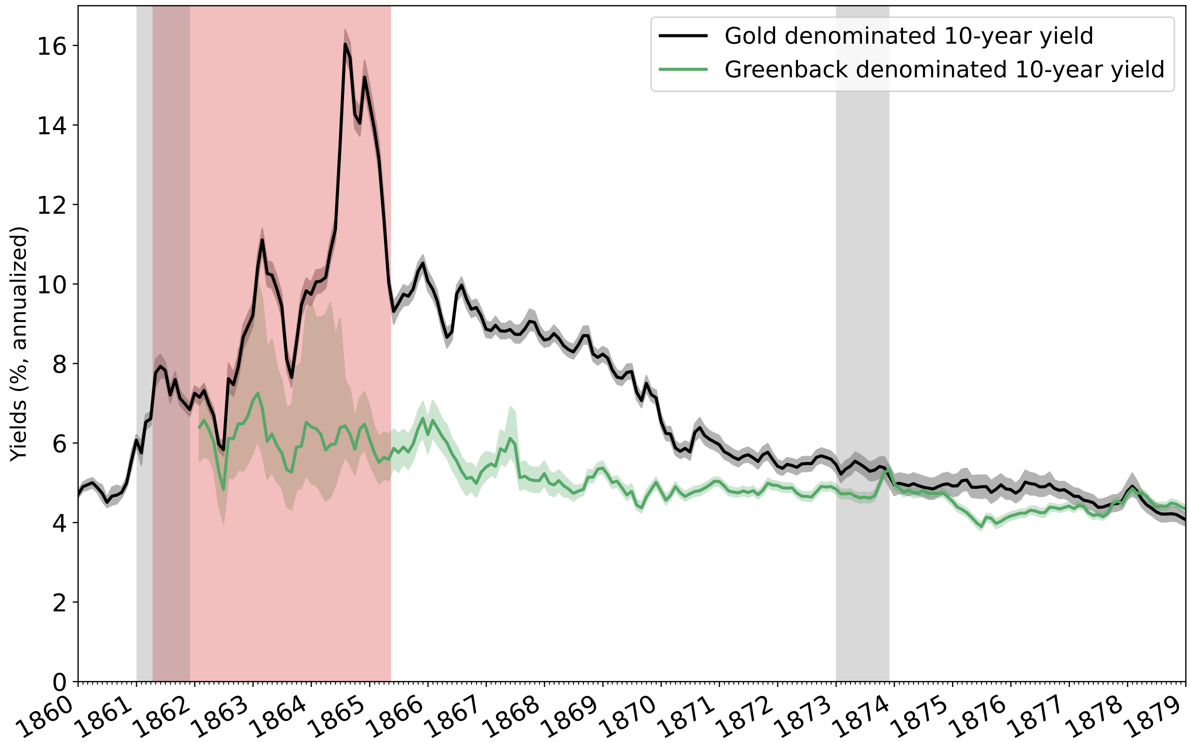


Figure VIII: Greenback Yields

The black line is the median of our posterior estimate for the 10-year gold denominated zero-coupon yield. The green line is the median of our posterior estimate of the 10-year greenback denominated zero-coupon yield. The bands around the posterior medians depict 90% interquantile ranges. The light gray intervals depict financial crises, and the light red interval depicts the Civil War.

greenback yields stayed flat at 6% throughout the Civil War reflecting that greenback bonds traded around par. Friedman and Schwartz (1963) (p. 69-70) write that:

> The behavior of interest rates in the United States is one of the most interesting features of the Civil War period and has puzzled most of its historians. . . . demand for loan funds must surely have been larger than any private demand that was suppressed by the diversion of resources to war use . . . Yet interest rates were unusually low . . . In our view, this is explained by speculative capital movements induced by the rise in the greenback price of gold.

By inferring how investors' expectations about the greenback-dollar exchange rate evolved during and after the Civil War, we can use our estimates to shed light on the hypothesis that anticipations of greenback appreciation kept greenback yields low.[26] The

[26] Roll (1972) also finds support for this hypothesis when he discusses the 1-year greenback yields

top panel of figure IX shows our estimates of Gold/Greenback exchange rates expected 2 and 10 years into the future at each date. The bottom panel shows the expected appreciation of greenbacks. Throughout the greenback era, investors evidently could have anticipated that greenback dollars would eventually exchange for gold dollars one-for-one. That kept long term greenback yields low. It is particularly striking that this was true even during the large drops in the value of the greenback that occurred in 1863 and 1864 in response to bad news from the war front. However, we also infer many fluctuations in investors' expectations about short term changes in greenback prices. These movements measure how political events after the war affected asset prices. Thus, we infer that the start of the Grant administration in 1869 and the Resumption Act of 1875 coincided with important shifts in investors' expectations, substantiating accounts by Calomiris (1988), Mitchell (1903), Mitchell (1908), and Studenski and Krooss (2003).

The years 1865-1869 saw widespread debates about retiring and redeeming greenback dollars. Congress started to retire greenbacks in April 1866 at the urging of Treasury Secretary Hugh McCullough but suspended that policy in February 1868 when the greenbacks became a contested issue in the 1868 presidential election. President Andrew Johnson and much of the Democratic Party wanted to reduce debt servicing costs by repaying principal in greenbacks. The Republican Party and its candidate, General Ulysses S. Grant, promised to service federal debts in gold dollars. Grant won. Ultimately, this led to an increase in the price of greenbacks, a decrease in expected greenback appreciation, and the closing of the gap between the greenback and gold 10-year yields.[27]

Progress toward establishing one-to-one convertibility temporarily stalled after the banking panic in 1873. In 1874, a bill initially legislating specie resumption by January 1, 1876 had mutated into a bill to create additional greenbacks, which President Grant vetoed. With the Resumption Act of January 14, 1875 that Congress reversed course and promised that the Treasury would stand ready to convert greenback dollars into gold dollars one-for-one starting on January 1, 1879. Our estimates of 2-year greenback-gold exchange rate expectations decreased throughout 1874 and 10-year expectations reached their lowest point. This indicates that by the early-1870s investors thought that discrepancies between gold and greenback prices would persist almost indefinitely and so gold and greenback yields converged.

Altogether, our results indicate widespread influences of the public's beliefs about prospective government policies.

during the Civil War.

[27]See Dewey (1902, Chapter XIV) for descriptions of these episodes.

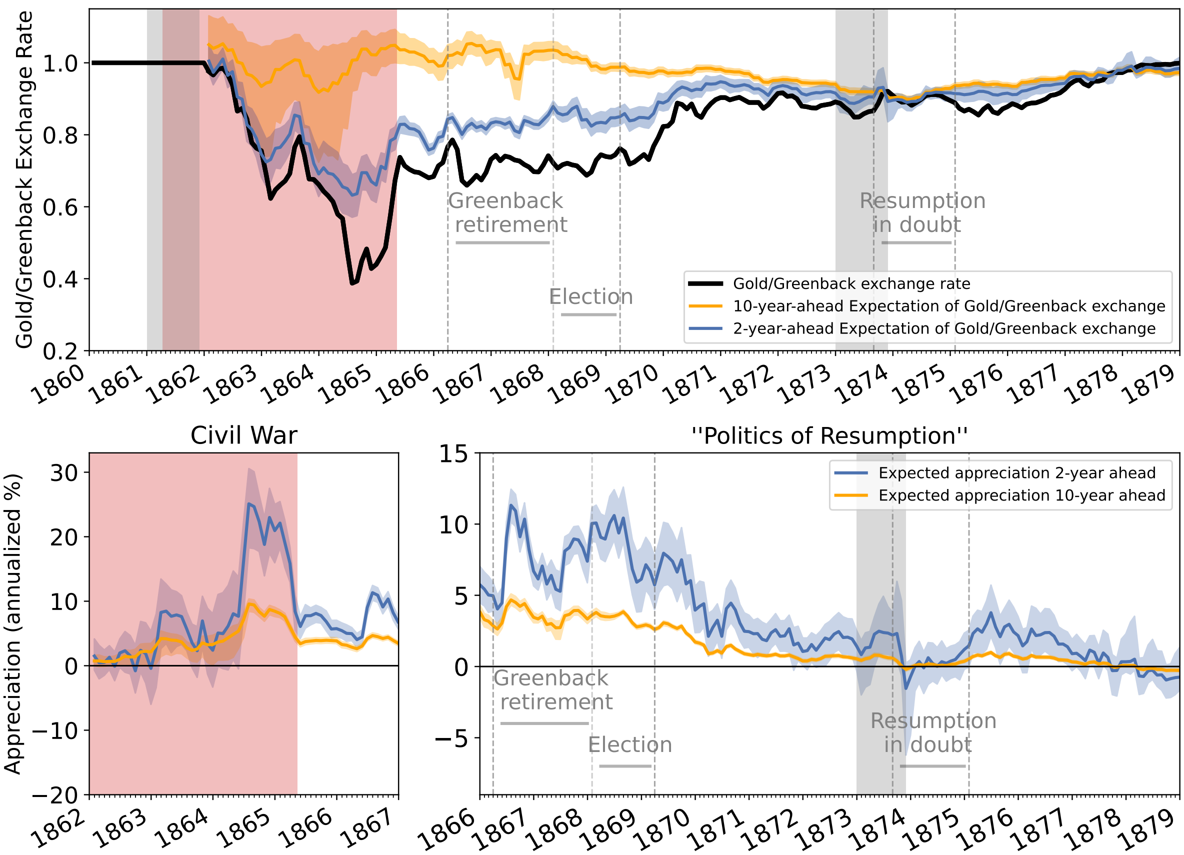


Figure IX: Nominal Anchor

*Top Plot:* The black line shows the path of the gold/greenback exchange rate, $P_t$. The orange line shows the median of our posterior estimate for the expected Gold/Greenback exchange rate 10 years into the future at each date. The blue line shows the median of our posterior estimate for the expected Gold/Greenback exchange rate 2 years into the future at each date. The shaded areas depict the 90% interquantile ranges for our estimate. *Bottom Plots:* Solid lines depict the median of our posterior estimates for the expected appreciation (yearly average) of the gold/greenback exchange rate over 2-year (in blue) and 10-year (in orange) horizons. The shaded areas depict the 90% interquantile ranges for our estimate. The period is broken into two subperiods, 1862-1867 and 1866-1879, to make magnitudes more visible. The light gray intervals depict financial crises, and the light red interval depicts the Civil War.

# 7 A "Short-Rate Disconnect"?

Recent analysts have inferred that today's US federal debt yield curves exhibit a "short-term disconnect," meaning that short term bonds are over-priced by pricing kernels that successfully price bonds at longer maturities.[28] In this section, we use our yield curves to infer that a similar short-rate-disconnect existed during some of the 19th century. We discuss how changes in short-rate-disconnects coincided with changing financial sector regulations.

## 7.1 Premium on short term bonds

Figure X shows our estimate of the "short-rate-disconnect" for the period 1800-1933. The pale blue dots depict differences between observed and model-implied yields to maturities for bonds with *less than one year* to maturity. Because our model was estimated using bonds with maturities greater than 1 year, these dots represent an "out-of-sample" extrapolations at the short end of the yield curve. The solid blue line depicts the 15-year centered moving average of these blue dots. The red solid line depicts the 15-year centered moving average of the difference between model-implied and observed yields-to-maturity for bonds with *more than one year* to maturity.

Evidently, differences between observed and model-implied yields to maturities average to about zero for bonds with long maturities but are systematically positive for extended periods for bonds close to maturity.[29] This means that a pricing kernel that successfully fits maturities greater than 1 year does not also fit maturities less than 1 year. This echoes the finding for modern data that Nelson and Siegel (1987) style parameterizations do not fit the short end of the modern yield curve well. Following the recent literature, we interpret this "short rate disconnect" as evidence for a liquidity premium on short term debt. By this, we mean that in a world without that liquidity premium, a pricing kernel that successfully fits long maturities would also fit short maturities. So, the "error" or "disconnect" in the fit of the Nelson and Siegel (1987) parameterization at the short end reflects the liquidity premium. We acknowledge other possible interpretations. However, we believe the concentration of errors very close to maturity zero, the systematic positive bias in the errors, and the tight fit away from zero make our interpretation sensible. We also believe that the extensive recent discussion of the modern short-rate-disconnect

[28] For instance, see Duffee (1996), Greenwood et al. (2018), and Lenel et al. (2019).

[29] As we saw in Figure II the differences also average out within a finer set of maturity bins. We take this as evidence that the short rate disconnect is not driven by the inability of the Nelson and Siegel (1987) parameterization to fit the maturities that we include in our estimation.

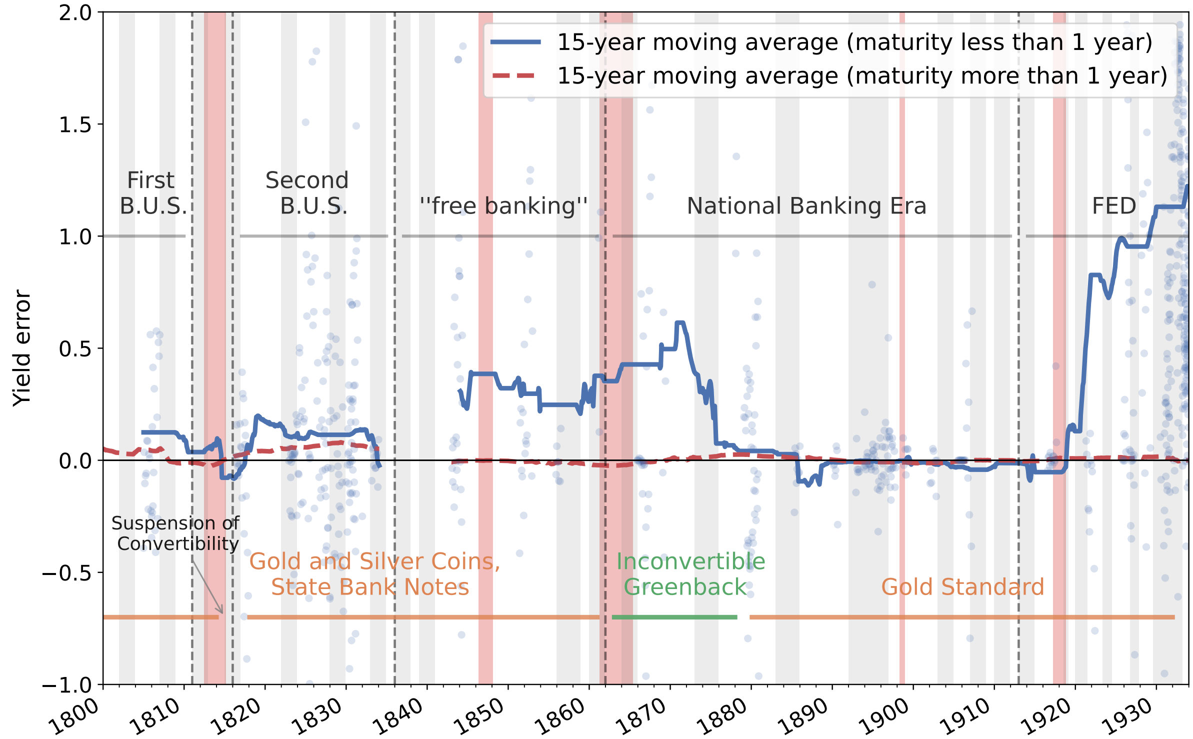


Figure X: Short Rate Disconnect

Pale blue dots depict the difference between model-implied and observed yield-to-maturities for bonds with *less than one year* to maturity. The solid blue line depicts the 15-year centered moving average of these dots excluding yield errors with a magnitude greater than 4 (to handle potential outliers from data issues). The red dashed line depicts the 15-year centered moving average of the difference between model-implied and observed yield-to-maturities for bonds with *more than one year* to maturity. The light gray intervals depict recessions, and the light red intervals depict wars.

makes it important to report the historical short-rate-disconnect, even though there are other ways of estimating liquidity premia. Under this interpretation, until the 1880s, bonds close to maturity traded at a premium in a range of 0.25 to 0.5 percentage points. The premium effectively disappeared from the 1880s until the First World War, before reappearing in the 1920s.

## 7.2 Re-assessing the National Banking Era

The evolution of the short rate disconnect seems to reflect interactions among monetary policies, financial regulations, and US borrowing costs.

*1791-1862: Bimetallism, Banks of The US, and State Banks.* At various times before the

Civil War, state legislatures chartered state banks that could issue demand notes backed by combinations of state debts and gold. Initially, the First (1791-1811) and Second (1816-1836) Banks of the United States operated at the national level with the special privileges of acting as banker for the federal government (depositing tax revenue and making loans) and operating across state boundaries. After President Jackson vetoed rechartering the Second Bank in 1932, some states issued charters to banks without requiring explicit approval from their legislatures, an arrangement sometimes called "free banking." Some state bank notes carried substantial and volatile discounts relative to gold dollars. The widespread circulation of such notes seems consistent with the persistent premium we inferred in the first half of the 19th century. Thus, the short rate disconnect was especially large early in the life of the Second Bank, which coincided with the Second Bank's actions to restrict state banks' money creation and credit before the financial crisis of 1819.

*1862-1913: Greenbacks, Gold Standard, and the National Banking System.* The outbreak of the Civil War in 1861 strained the Union's monetary and financial systems, provoking Congress to respond. In January 1862, state banks stopped converting their notes into gold dollars (they "suspended" convertibility). Congress passed National Banking Acts in 1862, 1863, 1865, and 1866 that established a system of national banks and the Office of the Comptroller of the Currency. National banks faced portfolio restrictions[30] and were allowed to issue bank notes up to 90% of the minimum of par and market value of qualifying US federal bonds.[31] The Congress wanted national bank notes to replace state bank notes and form a standardized national currency. Congress imposed a 10% annual tax on state bank notes, which was much larger than the annual tax rate on national bank notes of 1.0% before 1900 and 0.5% after 1900.

Key goals of the National Banking Acts were to increase supplies of liquid assets and to increase financial sector demand for long term US federal debt to help lower borrowing costs. In terms of our framework, the National Banking Acts sought to eliminate a short rate disconnect and decrease long term yields. Figure VI indicates that neither goal was achieved during the 1860s but that by the 1880s both goals had mostly been achieved.

[30] National banks could operate only one branch. They were restricted from making mortgages unless they were operating in rural areas, where they could make a limited range of loans collateralized by agricultural land.

[31] National banks could issue bank notes for circulation according to the following rules. Banks had to deposit particuar classes of US Treasury bonds as collateral for note issuance. Eligible bonds were US federal registered bonds bearing coupons of 5% or more. Deposited bonds had to be at least one-third of the bank's capital (not less than $30,000). Banks could issue bank notes up to an amount of 90% of the maximum of the market value of the bonds and the par value of the bonds. The 90% was changed to 100% in 1900.

These findings shed light on a long standing puzzle about National Bank note issues. Because yields on eligible treasuries were not low enough to compensate for the tax rate on notes outstanding, researchers have argued that there was persistent under-issuance of national bank notes during the National Banking Era (see Friedman and Schwartz (1963), Cagan (1965), Cagan and Schwartz (1991), Champ et al. (1994), Champ and Wallace (2003), Champ (2007), Calomiris and Mason (2008) for discussions of the puzzle). Our estimates confirm this situation: 2-year and 10-year yields typically exceeded the tax rate. However, as acknowledged by many researchers, because many other forces could have shifted levels of yield curves, comparing yields to the tax rate on note issuance might be a misleading way to infer effects of the National Banking Acts. Our analysis shows that if we focus on the short rate disconnect rather than the spread relative to the tax rate, then it appears that the National Banking Acts achieved considerable successes after the Treasury began pegging the exchange rate between greenback dollars and gold dollars in January 1879. This timing indicates that parity between greenbacks and gold dollars helped the National Banking Acts to work as intended, which in turn suggests that bank note issuance had initially been restrained by risks associated with currency devaluation, as hypothesized by Cagan (1965).

*1913-1933: The Federal Reserve System.* The Federal Reserve System (Fed) was created in 1913 to act as a reserve money creator of last resort. Figure X shows that a short-rate-disconnect reemerged soon after the Fed's creation, increasing sharply from around 0 to 1.0 percentage points in 1920. This likely reflects changes in how the Fed managed the market for Certificates of Indebtedness – interest-bearing securities with less than 1 year maturity – since most of the blue dots in Figure X during the Fed era correspond to these Certificates.

During World War I, the US Treasury used Certificates of Indebtedness to smooth mismatches between quarterly tax receipts and Liberty Bonds' irregular coupons and principal payments. Before 1920, these Certificates did not have a secondary market, but the Fed provided liquidity for banks via discount window lending secured by Certificates. However, after 1920, the New York Fed decided to create a secondary market for Certificates. That increased their liquidity relative to other government bonds. Garbade (2012, chapter 13) describes three innovations the New York Fed used to implement this change: (i) the Fed began raising discount rates enough to induce the Treasury to let certificates be traded below par, (ii) they extended credit to banks and dealers via repurchase agreements ("repos") secured by certificates, and (iii) they offered a service for "wire transfers" of certificates.[32]

Large variations in short rate disconnects across different regulatory regimes opens the possibilities for interpreting them as intended outcomes of regulatory changes.

# 8 Concluding Remarks

During the nineteenth century US Federal government debt went from being priced like a "junk bond" to being priced as a global "safe-asset". We have inferred patterns in US federal debt financing costs from 1791 to 1933 that shed light on the effectiveness of the financial and fiscal policy measures that accompanied this change.

From 1790 to 1829, US administrations took a sequence of actions that were designed to lower US borrowing costs. The federal government restructured Revolutionary War IOUs, established a gold dollar, restricted the states' ability to issue paper currency, and introduced the First and Second Banks of the United States to serve as fiscal agent of the federal government and to regulate state banks' creation of money and credit. We have described how these reforms reduced spreads between US and UK yields by the 1820s, indicating that federal policymakers seem to have succeeded in fostering a reputation for reliably servicing federal debts.

But these achievements did not endure. Disagreements about the Second Bank of the US culminated in Andrew Jackson's 1832 veto of a bill to recharter the Second Bank. After many states defaulted early in the 1840s, spreads between US and UK gold denominated yields returned to 1790s levels and then rose well above 1790s levels during the Civil War.

Difficulties in financing the Civil War persuaded the government to reorganize the monetary and financial systems in an attempt to lower Union borrowing costs. Between 1862 and 1878 the government issued non-convertible greenback dollars. Although greenbacks quickly depreciated relative to gold dollars, we infer that long run expectations of an appreciating greenback served as a heavy nominal anchor that allowed the Union to issue greenback-paying bonds at low greenback-denominated yields throughout the Civil War. Congress also established the National Banking System in four Acts between 1862 and 1866. This system promoted demands for longer-term federal debt by authorizing National Banks to issue bank-notes (i.e., paper currency) provided that they backed them by long-term federal bonds. Our estimates of a "short rate disconnect" constitute indirect evidence about how those policies affected the demand for long term debt. Relative to long-term debt, we find that short-term debt carried a money-like premium until the

[32] The US Treasury started issuing *zero-coupon* Treasury Bills in 1929.

1880s, but not between the 1880s and World War I. We interpret this pattern as providing evidence that the National Bank demand for long-term federal debt led to long-term debt being priced in similar ways with short-term debt.

During the 1880s, US yields approach UK yields. Although US yields again temporarily rose above UK yields in the 1890s, by 1900, US yields were actually well below UK yields, foreshadowing the eventual emergence of US debt as a global safe asset in the 20th century. Thus, by 1900 much of Alexander Hamilton's project to decrease US Federal borrowing costs had been completed.

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# A Data Appendix (For Online Publication)

Here we provide details about how the data sets were constructed. We first outline which bonds are included and excluded in the different estimation exercises. We then describe the assumptions made in constructing the cash flow series. Finally, we discuss the recession bands that we use. Some of these points have already been referenced in the main text, but we collect the assumptions here for completeness.

## A.1 Summary of Data Sources

We combined existing historical databases with transcription from the digital archives of newspapers and government reports. Table III summarizes the different data sources that we have used throughout the paper. The data set for bond prices and quantities is available at the Github repository `https://github.com/jepayne/US-Federal-Debt-Public` and construction methods are explained in Hall et al. (2018).[33] In this subsection of the appendix, we spotlight decisions about our data that we made to prepare for the statistical inferences presented in this paper.

Our bond price data are monthly. When available, we use the closing price at the end of each month. However, if a closing price is not available, then we use an average of high and low prices or an average of bid and ask prices. The sources for the price data from 1776 to 1839 are *Global Financial Data*, Razaghian (2002), and Sylla et al. (2006). Prices from 1840 to 1859 are from Razaghian (2002), *The New York Times*, and *Global Financial Data*. Prices from 1860 to 1925 are from the *Commercial and Financial Chronicle*, *Global Financial Data*, Martin (1886), *Merchants' Magazine and Commercial Review*, the *New York Times*, and *US Treasury Circulars*. When overlaps occurred, data were taken from the US Treasury Circulars. Prices from 1919 to 1925 are from "United States Govt. Bonds" tables in the *New York Times*. Prices after 1925 are taken from the *CRSP US Treasury Database*.[34] Data on contractually promised dollar payments come from Bayley (1882) for the period from 1790-1871 and from U.S. Department of the Treasury (2015) *Monthly Statements of the Public Debt* for the period from 1872-1960.

The quantity data are quarterly from 1776 to 1871 and monthly thereafter. All quantity entries record the quantity outstanding on the last business day of the period. The quantities outstanding from 1790 to 1871 are imputed from the issue and redemption series reported by Bayley (1882). We cross-checked these quantities against the quantity

[33] Only data from publicly available data sets are posted on the GitHub page.

[34] See `http://www.crsp.com/products/research-products/crsp-us-treasury-database`.

outstanding series reported in Register's Office (1886). After 1871 our source for quantity outstanding series is the U.S. Department of the Treasury (2015) *Monthly Statements of the Public Debt*. The call data are from *Annual Reports of the Secretary of Treasury* for various years. Data on Treasury securities held in government accounts are from *Banking and Monetary Statistics 1914-1941* prior to 1941 and from *Treasury Bulletin* thereafter.[35]

We require data on greenback-gold dollar exchange rates to estimate the greenback and real yield curves. For the gold-to-greenback exchange rate, we use Greenback price data from Mitchell (1908)[36] for the period from 1862-1878 during which greenbacks and gold dollars both circulated. For the gold-to-goods exchange rate, we combine several series. For the period from 1800-1860, we use the wholesale price index from Warren et al. (1932). For the period from 1860-1913, we use the General Price Level Index from the *NBER Macroeconomics Database*[37]. For the period from 1913-2020, we use the Consumer Price Index from the U.S. Bureau of Labor Statistics. Finally, we use the GDP series from Officer and Williamson (2021).

## A.2 Exclusion of Bonds

The estimation of the state space model of gold bond prices defined in Section 3.1 excludes all bonds that paid coupons and/or principals in any denomination other than gold. It also excludes short term Treasury notes that the US issued during the War of 1812. Bayley (1882) lists these notes as the Treasury Notes of 1812, Treasury Notes of 1813, Treasury Notes of March 1814, Treasury Notes of December 1814, and the Small Treasury Notes of 1815. These notes were used for payments well after their earliest redemption date and so probably earned a convenience yield. It also excludes the Panama Canal bonds, which we were not able to price consistently with the rest of the bonds, suggesting that they have a different pricing kernel.

For the estimation of the non-linear state space model of greenback bonds defined in Section 6.2 we exclude the following bonds, which Bayley (1882) documents had ambiguous denominations for the repayment of the principal: the "Five-Twenties of March 1864", the "Five-Twenties of June 1864", the "Five-Twenties of 1865", the "Consols of 1865", the "Consols of 1867", and the "Consols of 1868".

[35]See Board of Governors of the Federal Reserve System (1943) and Register's Office (1886).
[36]See Table 2
[37]See `https://www.nber.org/research/data/nber-macrohistory-iv-prices`

Table III: Summary of Data Sources

| Series | Period | Frequency | Source |
|---|---|---|---|
| Bond prices | 1776-1839 | M | Razaghian (2002), Sylla et al. (2006) and *Global Financial Data*. |
| | 1840-1859 | M | Razaghian (2002), *The New York Times*, and *Global Financial Data* |
| | 1860-1925 | M | *Commercial & Financial Chronicle*, *Global Financial Data*, *Merchant's Magazine*, *The New York Times*, *US Treasury Circulars*, and Martin (1886). |
| | 1925-1960 | M | *CRSP US Treasury Database*. |
| Quantities | 1790-1871 | Q | Bayley (1882). |
| | 1872-1960 | M | U.S. Department of the Treasury (2015). |
| Contract Info. | 1790-1960 | | 1790-1871 from Bayley (1882). |
| | | | 1872-1960 from U.S. Department of the Treasury (2015). |
| Gold/Goods | 1800-1860 | M | Wholesale Price Index (Warren/Pearson) |
| Exchange Rate | 1860-1913 | M | U.S. Index of the General Price Level (NBER Macrohistory: Series NBER 04051) |
| | 1913-2020 | M | CPI (BLS) |
| GDP | 1790-2020 | A | Officer and Williamson (2021) |
| Gold/Greenbacks | 1862-1878 | M | Yale SOM ICF dataset |
| Exchange Rate | | | |

[1] Repository for bond time series: `https://github.com/jepayne/US-Federal-Debt-Public`

## A.3 Construction of cash-flows

In order to estimate the yield curve, we need to construct the currency flows promised by each bond. For many of the early bonds in the sample, both the coupon dates and the maturity date have ambiguity because the bond information is imprecise and because it unclear whether newspaper prices are ex or cum dividend. For the coupon dates, we used the following rule. If Bayley (1882) lists exact coupon dates, then we use those dates. Otherwise, we identify the coupon dates from cyclical decreases in the price series at the frequency of coupon payment. We interpret these decreases as the price impact of the coupon payment.

For the maturity dates, we used the following rules. For bonds with an explicit maturity date, we set the maturity to that date. For the three Hamilton bonds (which Bayley (1882) lists as *Six Per Cent Stock of 1791*, *Deferred Six Per Cent Stock of 1791*, and the *Three Per Cent Stock of 1791*), which were issued as annuities but ultimately redeemed early, we impose that investors had perfect foresight about the early redemption and set the maturity date to be the date at which greater than 90% of the outstanding bonds had been redeemed. For bonds with a redemption window, we calculate the minimum of the date at which 90% of the outstanding bonds had been redeemed and date at which the bonds started to trade at par value. We then set the maturity date to be the closest coupon payment date to that minimum calculation. For bonds that converted into different bonds, we set the maturity date to be the maturity of the bond into which it is converted.

## A.4 Construction of Recession bands

For the 1796-1914 period we use recession dates from Davis (2006). These are derived solely from the Davis (2004) annual industrial production index. The Davis index incorporates 43 annual series in the manufacturing and mining industries in a manner similar to the Federal Reserve Board's present-day industrial production index. For this reason, we regard it as an improvement over earlier more qualitative approaches of dating pre-World War I business cycles. Since the data used to date peaks and troughs is annual, the methodology is quite simple: A year immediately preceding an absolute decline in the aggregate level of Davis's industrial production index defines a peak, and the last consecutive decline following a peak defines a trough Davis (2006). For the 1915-present period we use recession dates from the NBER.

# B Technical Details (For Online Publication)

In this online appendix we outline the technical details behind the estimation of this paper. In our companion paper, Payne et al. (2023a), we compare our procedure to other possible statistical approaches and discuss its advantages for handing historical data sets.

## B.1 Model of Gold Yield Curves

We use three pieces of information on each bond included in our sample: (1) the observed period-$t$ price of bond $i$ in terms of gold, $\tilde{p}_t^{(i)}$, (2) the sequence of promised gold dollar coupon and principal payments of bond $i$ at time $t$, $\{m_{t+j}^{(i)}\}_{j=1}^{\infty}$, and (3) the Macaulay duration of bond $i$ in period $t$ denoted by $d_t^{(i)}$. Collect these objects in a tuple

$$\mathcal{D}_t := \left\{\left(\tilde{p}_t^{(i)}, \mathsf{m}_t^{(i)}, d_t^{(i)}\right)\right\}_{i\in\mathcal{M}_t}$$

where $\mathcal{M}_t$ denotes the set of bonds available at period $t$. The time series of $\mathcal{D}_t$ constitute our data. For simplicity, let $x^T := \{x_t\}_{1\leq t\leq T}$ for any sequence $x$, then our data can be written as $\mathcal{D}^T$. The statistical model that we fit to this data to obtain a gold dollar yield curve can be framed as the following nonlinear state space model:

$$\begin{aligned}
\tilde{p}_t^{(i)} &= \sum_{j=1}^{\infty} q_t^{(j)} m_{t+j}^{(i)} + d_t^{(i)} \sigma_m^{(i)} \varepsilon_t^{(i)} && \text{observation eq [bond prices]} \\
\lambda_{t+1} &= \lambda_t + \Sigma_t^{\frac{1}{2}} \varepsilon_{\lambda,t+1} && \text{state eq [yield curve parameters]} \\
\log\sigma_{t+1} &= \log\sigma_t + \Xi_\sigma \varepsilon_{\sigma,t+1} && \text{state eq [stochastic volatility]} \\
\text{with} \quad & \varepsilon_t^{(i)} \sim \mathcal{N}(0,1) \ \ \forall i, \quad \varepsilon_{\lambda,t} \sim \mathcal{N}\left(\mathbf{0}, \mathbb{I}_3\right) \quad \varepsilon_{\sigma,t} \sim \mathcal{N}\left(\mathbf{0}, \mathbb{I}_3\right), \forall t \geq 1
\end{aligned}$$

where $\lambda_t := [L_t, S_t, C_t]'$ is a vector of three latent factors: level, slope, and curvature, respectively. The period-$t$ discount function, $\{q_t^{(j)}\}_{j=1}^{\infty}$, can be written as a function of these factors and location parameter $\tau$:

$$q_t^{(j)} = \exp\left(-\frac{1}{1200j}\left[L_t + S_t\left(\frac{1-\exp(-j\tau)}{j\tau}\right) + C_t\left(\frac{1-\exp(-j\tau)}{j\tau} - \exp(-j\tau)\right)\right]\right) \quad \text{(B.1)}$$

where the division by 1200 ensures that the zero-coupon yields are expressed in terms of annual percentage points.

We decompose the covariance matrix of the shocks to $\lambda_t$ as $\Sigma_t = \Xi_t \Omega \Xi_t$, i.e., we write it as the product of the time-invariant correlation matrix, $\Omega$, and the time-varying volatilities, $\Xi_t := \operatorname{diag}(\sigma_t)$ where $\sigma_t := [\sigma_{1,t}, \sigma_{2,t}, \sigma_{3,t}]'$. This decomposition implies $\Sigma_t^{\frac{1}{2}} =$

$\Xi_t L\Omega$ where $L\Omega$ denotes the (lower-triangular) Cholesky factor of $\Omega$. Finally, we assume that shocks to $\sigma_t$ are independent, so $\Xi_\sigma$ is a 3-by-3 diagonal matrix.

The latent states for this system are the sequences $\{\lambda_t\}_{t\geq 1}$ and $\{\sigma_t\}_{t\geq 1}$ with initial conditions $\lambda_0 = [L_0, S_0, C_0]'$ and $\sigma_0$. The total parameters for this system are:

$$\vartheta = \left(\tau,\ \{\sigma_m^{(i)}\}_{i\in\mathcal{I}},\ L_0,\ S_0,\ C_0,\ \sigma_0,\ \Omega,\ \Xi_\sigma\right)$$

where $\mathcal{I} := \bigcup_{t=1}^{T} \mathcal{M}_t$ denotes the universe of gold paying bonds in our sample.

We estimate latent states $\{\lambda_t\}_{t\geq 1}$ and $\{\sigma_t\}_{t\geq 1}$ and parameters $\vartheta$ using Bayesian methods. The log-likelihood can be constructed by adding up Gaussian log-likelihoods associated with the independent shocks:

$$\log f\left(\mathcal{D}^T|\lambda^T, \sigma^T, \vartheta\right) = -\frac{1}{2}\sum_{t=1}^{T}\left(\log|\Sigma_t| + \Delta\lambda_t'\Sigma_t^{-1}\Delta\lambda_t\right) - T\left(\log|\Xi_\sigma| + \frac{1}{2}\sum_{t=1}^{T}\Delta\sigma_t'\Xi_\sigma^{-2}\Delta\sigma_t\right)$$

$$-\sum_{t=1}^{T}\left(\sum_{i\in\mathcal{M}_t}\log|\sigma_m^{(i)}| + \frac{1}{2}\Delta\tilde{p}_t'\Xi_{\mathcal{M}_t}^{-2}\Delta\tilde{p}_t\right)$$

where $f$ is a conditional density, $\Delta\lambda_t := \lambda_t - \lambda_{t-1}$ and $\Delta\sigma_t := \sigma_t - \sigma_{t-1}$, and $\Xi_{\mathcal{M}_t}^2$ is the covariance matrix of the corresponding (inverse-duration-weighted) pricing errors:

$$\Delta\tilde{p}_t := \left\{\frac{\tilde{p}_t^{(i)} - \sum_{j=1}^{\infty} q_t^{(j)} m_{t+j}^{(i)}}{d_t^{(i)}}\right\}_{i\in\mathcal{M}_t}$$

We obtain the posterior by combining this likelihood function with priors for $\vartheta$, which are specified in Table I.[38] The priors for $\vartheta$ induce a prior distribution for the latent states $(\lambda^T, \sigma^T)$ through the model's state equations. We denote this combined prior distribution as $f(\lambda^T, \sigma^T, \vartheta)$. We use a log-normal prior for $\tau$ and independent normal priors for the three entries of the initial $\lambda_0$ vector that implies a flat "average yield curve", i.e., for each $t$ and $j$ the prior mean of $y_t^{(j)}$ is 10% with standard deviation of around 5%. In addition, we use independent log-normal priors for the three elements of the initial $\sigma_0$ vector and independent exponential priors for the set of pricing errors, $\{\sigma_m^{(i)}\}_{i\in\mathcal{I}}$, and the volatilities of the $\sigma_t$ shocks.

[38] We specify weakly informative prior distributions for the model's hyper-parameters for the specific purpose of *regularizing* our estimator and facilitating smooth operation of the sampling algorithm.

The log posterior of our nonlinear state space model is given by:

$$\log f\left(\lambda^T, \sigma^T, \vartheta \mid \mathcal{D}^T\right) = \log f\left(\mathcal{D}^T \mid \lambda^T, \sigma^T, \vartheta\right) + \log f(\lambda^T, \sigma^T, \vartheta)$$

We approximate posterior probabilities by deploying the Hamiltonian Monte Carlo and No U-Turn sampler (HMC-NUTS).

Computational issues: While `Stan` might seem an obvious choice for the task at hand—it is a well-developed software that efficiently implements the HMC-NUTS sampler—non-trivial features of our data set make it inconvenient for our purposes. Some of our main technical difficulties are: (1) the number of observed assets changes over time, (2) each bond has a payoff stream of varying length, (3) there are many periods without price observations, (4) the set of bond-specific pricing errors that are relevant at a given period $t$ changes over time in a complicated fashion, etc. To tackle these difficulties, we code the log posterior function of our model from scratch and feed it into the DynamicHMC.jl package by Papp et al. (2021) which is a robust implementation of the HMC-NUTS sampler mimicking many aspects of `Stan`. An important advantage of this package is that it allows the user to provide the Jacobian of the log-posterior manually. Not relying on automatic differentiation cuts running time by several orders of magnitude. In most cases, we use the recommended (default) tuning parameters for the NUTS algorithm.

Importance of Pooling: While the original motivation behind the Dynamic Nelson-Siegel model was to provide good yield curve forecasts, we are primarily interested in its attractive information pooling properties. To better appreciate the benefits of flexibly modeling the time-dependence between the elements of $\{\lambda_t\}_{t\geq 1}$ in the presence of bond-specific pricing errors, Figure XI depicts estimated gold denominated yields that come from a model *without* these features. In particular, for each period $t$—for which we have at least four observed bond prices—we estimated the four yield curve parameters $(\lambda, \tau)$ by nonlinear least squares, i.e., by solving the following minimization:

$$\min_{\lambda,\tau} \sum_{i\in\mathcal{M}_t} \frac{\tilde{p}_t^{(i)} - \sum_{j=1}^{\infty} q^{(j)}(\lambda,\tau) m_{t+j}^{(i)}}{d_t^{(i)}} \tag{B.2}$$

which follows closely the common approach adopted in modern contexts (see e.g. Gürkaynak et al. (2007)).

The solid lines in Figure XI depict the 2-year and 10-year zero-coupon yield estimates

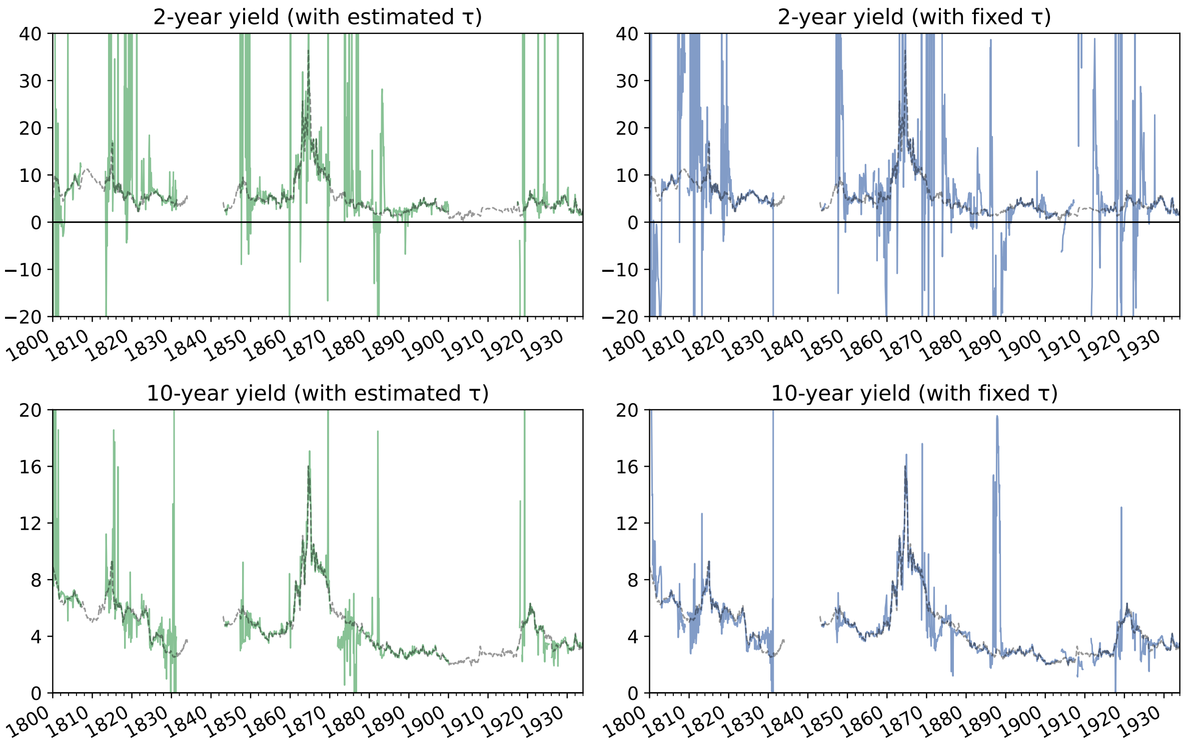


Figure XI: Month-by-month Estimation with Non-Linear Least Squares

The green solid lines on the left panels depict the 2-year (top) and 10-year (bottom) zero-coupon yields estimated by nonlinear least squares month-by-month. These estimates were computed by minimizing with respect to the four parameters of the Nelson-Siegel model. The blue solid lines on the right panels depict the 2-year (top) and 10-year (bottom) zero-coupon yields estimated by nonlinear least squares month-by-month assuming that the location parameter is fixed at $\tau = 60$. The dashed grey lines depict our baseline gold denominated zero-coupon yield estimates (posterior median). The y-axes are restricted to show a "reasonable" range of yields (in terms of annualized percentage points). Least squares estimates of the 2-year zero-coupon yield often reach values around 2,000%.

without information pooling, i.e., using the month-by-month non-linear least squares estimates. For comparison, the dashed grey lines show our baseline yield estimates (posterior median). The green lines on the left panels correspond to the case when all four parameters (including $\tau$) of the Nelson-Siegel model are estimated. The right panels show the case when we fix $\tau = 60$ (close to our posterior median estimate) and estimate only the level, slope, and curvature parameters.

The least squares estimates, especially at the short end, exhibit excessive variation showing clear signs of over-fitting. The gaps that are present in the green and blue lines represent periods when either the number of bonds is lower than the number of parameters to be estimated, or the numerical optimizer struggles to identify a (local) minimium of (B.2). Comparing the left panels to the right panels illustrates that fixing the location parameter $\tau$ helps somewhat to obtain more sensible yield curves and for more periods (especially for longer-term yields), but finding the "right" $\tau$ is a difficult task and is one of the main reasons why we need an estimator in the first place.

## B.2 Model of Exchange Rates And Greenback Yield Curves

For the period between 1862 and 1879, we jointly estimate the greenback yield curve and the dynamic relationship between two time series: the gold-to-greenback exchange rate, $P_t$, and the gold-to-goods exchange rate, $G_t$. For simplicity, we collect the exchange rates in a vector $v_t = [P_t, G_t]'$. In addition, we use three pieces of information on each greenback paying bond in our sample: (1) the observed period-$t$ gold dollar price of greenback paying bond $i$, $\tilde{p}_t^{(i,g)}$, (2) the sequence of promised greenback dollar coupon and principal payments of bond $i$ at time $t$, $\{m_{t+j}^{(i,g)}\}_{j=1}^{\infty}$, and (3) the Macaulay duration of bond $i$ in period $t$ denoted by $d_t^{(i,g)}$. Collect these objects in a tuple

$$\mathcal{D}_t := \left( v_t, \ \left\{ \left( \tilde{p}_t^{(i,g)}, \mathsf{m}_t^{(i,g)}, d_t^{(i,g)} \right) \right\}_{i \in \mathcal{M}_t^{(g)}} \right)$$

where $\mathcal{M}_t^{(g)}$ denotes the set of greenback paying bonds available at period $t$. The time series $\mathcal{D}^T$ constitutes our data. The statistical model that we fit to this data is:

$$\begin{aligned}
\tilde{p}_t^{(i,g)} &= \frac{1}{(e_1' v_t)} \sum_{j=1}^{\infty} q_t^{(j)} \mathbb{E}_t[(e_1' v_{t+j})] m_{t+j}^{(i,g)} + d_t^{(i,g)} \sigma_m^{(i)} \varepsilon_t^{(i)} & \varepsilon_t^{(i)} &\sim \mathcal{N}(0,1), \ \forall i \\
v_{t+1} &= \mu_t + A_{1,t} (v_t - \mu_t) + A_{2,t} (v_{t-1} - \mu_t) + \varepsilon_{v,t+1} & \varepsilon_{v,t} &\sim \mathcal{N}(\mathbf{0}, \Sigma_v) \\
\mu_{t+1} &= \mu_t + \Xi_\mu \varepsilon_{\mu,t+1} & \varepsilon_{\mu,t} &\sim \mathcal{N}(\mathbf{0}, \mathbb{I}_2) \\
\overrightarrow{\mathbf{A}}_{t+1} &= \overrightarrow{\mathbf{A}}_t + \Xi_A \varepsilon_{A,t+1} & \varepsilon_{A,t} &\sim \mathcal{N}(\mathbf{0}, \mathbb{I}_8)
\end{aligned}$$

where $\overrightarrow{\mathbf{A}}_t := [\text{vec}(A_{1,t})', \text{vec}(A_{2,t})']'$ is an 8-vector made up of the vectorized autoregression matrices, $e_1$ is a selector vector which selects the first entry of a vector, hence $e_1'v_t = P_t$. The conditional expectation in the bond pricing formula is given by

$$\mathbb{E}_t[e_1'v_{t+j}] = \mathbb{E}_t[P_{t+j}] = e_1'\left(\mathbf{m}_t + \mathbf{A}_t^j\mathbf{x}_t\right)$$

$$\text{where} \qquad \mathbf{m}_t := \begin{pmatrix}\mu_t \\ \mathbf{0}_2\end{pmatrix} \qquad \mathbf{A}_t := \begin{bmatrix}A_{1,t} & A_{2,t} \\ \mathbb{I}_2 & \mathbf{0}_{2\times 2}\end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_t := \begin{pmatrix}v_t - \mu_t \\ v_{t-1} - \mu_t\end{pmatrix}$$

We decompose the residual covariance matrix as $\Sigma_v = \Xi_v\Omega_v\Xi_v$, i.e., we write it as the product of the correlation matrix, $\Omega_v$, and the volatilities, $\Xi_v$, both being $2\times 2$ matrices.

In order to evaluate the greenback bond pricing formula, we need to assign values for the gold discount function $\mathsf{q}_t$ for each $t$. We use our Nelson-Siegel parameterization of the gold discount function, which expresses $\mathsf{q}_t$ as a function of $\lambda_t$ and $\tau$ (see Appendix B.1), and assume a multivariate Gaussian prior distribution for the sequence $(\{\lambda_t\}, \tau) \sim \mathcal{N}(\mu_{\mathsf{q}}, \Sigma_{\mathsf{q}})$. Assuming that the gold yield curve parameters $(\{\lambda_t\}, \tau)$ are normally distributed gives rise to an implied prior distribution for $\mathsf{q}_t$ via equation (B.1).

The latent states to be estimated are the sequences $\{\mu_t\}_{t\geq 1}$ and $\left\{\overrightarrow{\mathbf{A}}_t\right\}_{t\geq 1}$ with initial conditions $\mu_0$ and $\overrightarrow{\mathbf{A}}_0$. The total parameters to be estimated are:

$$\varphi = \left(\{\sigma_m^{(i)}\}_{i\in\mathcal{I}^{(g)}},\ \mu_0,\ A_{1,0},\ A_{2,0},\ \Omega_v,\ \Xi_v,\ \Xi_\mu,\ \Xi_A,\ \{\lambda_t\},\ \tau\right)$$

where $\mathcal{I}^{(g)}$ denotes the set of greenback paying bonds in our sample.

We estimate latent states $\{\mu_t\}_{t\geq 1}$ and $\left\{\overrightarrow{\mathbf{A}}_t\right\}_{t\geq 1}$ and parameters $\varphi$ by deploying the Hamiltonian Monte Carlo and No U-Turn sampler (HMC-NUTS). The log-likelihood can be constructed by adding up Gaussian log-likelihoods associated with the independent shocks:

$$\begin{aligned}
&\log f\left(\mathcal{D}^T|\mu^T, \overrightarrow{\mathbf{A}}^T, \vartheta\right) \\
&= -T\left(\log|\Xi_\mu| + \frac{1}{2}\sum_{t=1}^{T}\Delta\mu_t'\Xi_\mu^{-2}\Delta\mu_t\right) - T\left(\log|\Xi_A| + \frac{1}{2}\sum_{t=1}^{T}\Delta\overrightarrow{\mathbf{A}}_t'\Xi_A^{-2}\Delta\overrightarrow{\mathbf{A}}_t\right) \\
&\quad - \frac{T}{2}\left(\log|\Sigma_v| + \sum_{t=1}^{T}\Delta v_t'\Sigma_v^{-1}\Delta v_t\right) - \sum_{t=1}^{T}\left(\sum_{i\in\mathcal{M}_t}\log|\sigma_m^{(i)}| + \frac{1}{2}\Delta\tilde{p}_t^{(g)'}\Xi_{\mathcal{M}_t}^{-2}\Delta\tilde{p}_t^{(g)}\right)
\end{aligned}$$

where where $f$ is a conditional density, $\Delta v_t$ is the VAR forecast error, $\Delta\mu_t := \mu_t - \mu_{t-1}$ and $\Delta\overrightarrow{\mathbf{A}}_t := \overrightarrow{\mathbf{A}}_t - \overrightarrow{\mathbf{A}}_{t-1}$, and $\mathcal{M}_t$ denotes the set of available greenback bonds at period $t$, and $\Xi^2_{\mathcal{M}_t}$ is the covariance matrix of the corresponding (inverse-duration-weighted) pricing errors:

$$\Delta\tilde{p}_t^{(g)} := \left\{ \frac{\tilde{p}_t^{(i,g)} - \sum_{j=1}^{\infty} q_t^{(j)} z_t^{(j)} m_{t+j}^{(i,g)}}{d_t^{(i,g)}} \right\}_{i\in\mathcal{M}_t}$$

We obtain the posterior by combining this likelihood function with priors for $\varphi$, as described in Table II. We denote this combined prior distribution as $f(\mu^T, \overrightarrow{\mathbf{A}}^T, \vartheta)$. The only non-standard step is the way we choose the prior hyper-parameters of the gold discount function $\mathsf{q}_t$. In order to account for our uncertainty about the time series of gold yield curves, we assume that $(\{\lambda_t\}, \tau) \sim \mathcal{N}(\mu_{\mathsf{q}}, \Sigma_{\mathsf{q}})$ and use the posterior MCMC sample of the Section 3 estimation to compute the "posterior sample mean" and "posterior sample covariance matrix" of $(\{\lambda_t\}, \tau)$ and set $(\mu_{\mathsf{q}}, \Sigma_{\mathsf{q}})$ equal to these posterior sample statistics.

The log posterior of our nonlinear state space model is given by:

$$\log f\left(\mu^T, \overrightarrow{\mathbf{A}}^T, \vartheta \mid \mathcal{D}^T\right) = \log f\left(\mathcal{D}^T \mid \mu^T, \overrightarrow{\mathbf{A}}^T, \vartheta\right) + \log f(\mu^T, \overrightarrow{\mathbf{A}}^T, \vartheta)$$

We approximate posterior probabilities by deploying the Hamiltonian Monte Carlo and No U-Turn sampler (HMC-NUTS).